ISRAEL JOURNAL OF MATHEMATICS **148** (2005), 137–168

A CLASS OF MULTIPLIERS FOR $\mathcal{W}^\perp$

BY

ALEXANDRE I. DANILENKO*

Max Planck Institute for Mathematics, Vivatsgasse 7, D-53111 Bonn, Germany
and
Division of Mathematics
Institute for Low Temperature Physics & Engineering of
Ukrainian National Academy of Sciences
47 Lenin Ave., Kharkov, 61103, Ukraine
e-mail: danilenko@ilt.kharkov.ua

AND

MARIUSZ LEMAŃCZYK**

Faculty of Mathematics and Computer Science, Nicolaus Copernicus University
ul. Chopina 12/18, 87-100 Toruń, Poland
e-mail: mlem@mat.uni.torun.pl

Dedicated to Hillel Furstenberg on the occasion of his retirement

ABSTRACT

Let $\mathcal{W}^\perp$ denote the class of ergodic probability preserving transformations which are disjoint from every weakly mixing system. Let $\mathcal{M}(\mathcal{W}^\perp)$ be the class of multipliers for $\mathcal{W}^\perp$, i.e. ergodic transformations whose all ergodic joinings with any element of $\mathcal{W}^\perp$ are also in $\mathcal{W}^\perp$. Fix an ergodic rotation T, a mildly mixing action S of a locally compact second countable group G and an ergodic cocycle ϕ for T with values in G. The main result of the paper is a sufficient (and also necessary by [LeP] when G is countable Abelian and S is Bernoullian) condition for the skew product build from T, ϕ and S to be an element of $\mathcal{M}(\mathcal{W}^\perp)$. Moreover, the self-joinings of such extensions of T are described with an application to study semisimple extensions of rotations.

* The first-named author was supported in part by CRDF, grant UM1-2546-KH-03.
** The second-named author was supported in part by KBN grant 1P03A 03826.
Received September 17, 2003 and in revised form July 26, 2004

0. Introduction

In 1967, H. Furstenberg introduced a concept of disjointness for ergodic transformations as a sort of "extreme nonsimilarity" for them [Fu1]. In particular, disjoint transformations are nonisomorphic and even more, they have no nontrivial common factors. A nontrivial problem coming from [Fu1] is to describe the class $\mathcal{W}^\perp$ of those ergodic transformations that are disjoint with every weakly mixing transformation. It was actually shown there that $\mathcal{W}^\perp$ includes the class $\mathcal{D}$ of distal transformations. The fact that this inclusion is proper was established only in 1989 by E. Glasner and B. Weiss [GlW]. Later, E. Glasner introduced a class $\mathcal{M}(\mathcal{W}^\perp)$ of multipliers for $\mathcal{W}^\perp$, i.e., the class of transformations whose all ergodic joinings with any member of $\mathcal{W}^\perp$ are also in $\mathcal{W}^\perp$. We then have $\mathcal{D} \subset \mathcal{M}(\mathcal{W}^\perp) \subset \mathcal{W}^\perp$. Elaborating the ideas from [GlW], E. Glasner demonstrated that $\mathcal{D} \neq \mathcal{M}(\mathcal{W}^\perp)$ [Gl1]. Finally, in a recent paper of F. Parreau and the second-named author [LeP] it was shown that $\mathcal{M}(\mathcal{W}^\perp) \neq \mathcal{W}^\perp$. We now give some details on the latter result. Let T be an ergodic measure preserving transformation of a standard probability space $(X, \mathfrak{B}_X, \mu)$, $S = (S_g)_{g\in G}$ a measure preserving action of a locally compact second countable (l.c.s.c.) group G on a standard probability space $(Y, \mathfrak{B}_Y, \nu)$ and ϕ: $X \to G$ a Borel map. Throughout the paper we assume that G is not compact. Define two measure preserving transformations T_ϕ and $T_{\phi,S}$ of the product spaces $(X \times G, \mu \times \lambda_G)$ and $(X \times Y, \mu \times \nu)$ respectively by setting

$$T_\phi(x, g) = (Tx, \phi(x)g) \quad \text{and} \quad T_{\phi,S}(x, g) = (Tx, S_{\phi(x)}y),$$

where λ_G stands for a left Haar measure on G. Note that T_ϕ is infinite measure preserving. The following result was proved in [LeP]: if $T \in \mathcal{W}^\perp$, G is countable Abelian, S Bernoullian, ϕ is ergodic (i.e., T_ϕ is ergodic) and the group $e(T_\phi) \subset \mathbb{T}$ of $L^\infty(X \times G, \mu \times \lambda_G)$-eigenvalues of T_ϕ is uncountable, then $T_{\phi,S} \in \mathcal{W}^\perp$ and for any weakly mixing transformation R whose (reduced) maximal spectral type does not vanish on $e(T_\phi)$ there exists an ergodic self-joining η of $T_{\phi,S}$ such that $(T_{\phi,S} \times T_{\phi,S}, \eta)$ is not disjoint from R. In this connection a question arises: what happens if $e(T_\phi)$ is countable? The answer is the main result of the paper (see Section 8):

THEOREM 0.1: *Let T be an ergodic transformation with pure point spectrum and let G be an amenable l.c.s.c. group without nontrivial compact normal subgroups. Assume that S is a mildly mixing action of G. If ϕ: $X \to G$ is an ergodic cocycle of T for which $e(T_\phi)$ is countable then $T_{\phi,S}$ belongs to $\mathcal{M}(\mathcal{W}^\perp)$.*

This finally explains a relationship between Glasner–Weiss' generic techniques and our construction. Actually, we show that the set of ergodic cocycles with $e(T_\phi)$ countable is generic in the Polish space of all measurable maps from X to G. Moreover, the same is true for the subspace Φ_0 of continuous zero mean $\mathbb{R}$-valued cocycles of any irrational rotation (Φ_0 is furnished with the topology of uniform convergence). Taking any horocycle flow as S we then get as a corollary an extension of the main result from [Gl1]: $T_{\phi,S} \in \mathcal{M}(\mathcal{W}^{\perp}) \setminus \mathcal{D}$ for a generic ϕ from Φ_0 (there were some further restrictions on S and the rotation in [Gl1]).

Moreover, we obtain a full description of possible ergodic self-joinings of $T_{\phi,S}$ (under the assumptions of Theorem 0.1). This problem was already examined in [LMN] for Abelian G and some ergodic cocycles ϕ with the property that $\phi \times \phi \circ R$ is regular for each transformation R commuting with T. In this paper we make a step forward and analyze the general case of G and ergodic ϕ (but S is still mildly mixing). The case $\phi \times \phi \circ R$ is regular is treated similarly to the Abelian one. However, quite surprisingly it turns out that the case of nonregular $\phi \times \phi \circ R$ is easily handled due to a special property of its Mackey $G \times G$-action. In fact, the relatively independent extension of the graph joining μ_R is the only extension of μ_R to a self-joining of $T_{\phi,S}$ (see Theorem 7.3 for the precise statement).

Thus, as appears, the description of self-joinings of $T_{\phi,S}$ is very similar to what we have in the classical case of compact G (cf. [LeM], [Me]). As an application, we extend the main result of [LMN]:

THEOREM 0.2: *Let T, G, S satisfy the assumptions of Theorem 0.1. If S is in addition 2-fold-extra-simple (i.e., for each continuous group automorphism θ of G, every ergodic joining of S and $S \circ \theta$ is either the product measure or a graph-joining), then $T_{\phi,S}$ is semisimple and the extension $T_{\phi,S} \to T$ is relatively weakly mixing for every ergodic cocycle ϕ: $X \to G$.*

Notice that in the present paper we bypass the use of the spectral theory which played a crucial role in [LeL], [LeP] and [LMN]. That enables us to get rid of the commutativity assumption on G which was standing in those papers.

Finally, we would like to note that even though $T_{\phi,S} \to T$ seems to be a very special case of a general extension (see a theorem of L. Abramov and V. Rokhlin [AbR]), however one of our first observations is that each Rokhlin cocycle is cohomologous to a "locally compact" one. In other words, each extension is isomorphic as extension to one of the form $T_{\phi,S} \to T$. In fact, G can be taken as countable and amenable (see Proposition 2.1).

The outline of the paper is as follows. Section 1 contains a background on nonsingular group actions, joinings and measurable orbit theory. In Section 2 we show that any extension can be given by an amenable countable group action. Sections 3–6 are of technical nature. Group self-joinings and their connection with type I actions are considered in Section 3. Some specific properties of the Mackey actions associated to $\phi \times \phi \circ R$ are discussed in Section 4. In Section 5 we introduce a concept of relatively finite measure preserving extensions and investigate their properties. A useful link between some simplices of invariant and quasi-invariant measures is discussed in Section 6. The main results of the paper are collected in Sections 7–9: the ergodic self-joinings of $T_{\phi,S}$ are described in Section 7, the theorem on multipliers for $\mathcal{W}^{\perp}$ is proved in Section 8 and semisimplicity of $T_{\phi,S}$ is studied in the final Section 9.

ACKNOWLEDGEMENT: The first-named author would like to thank N. Copernicus University for the warm hospitality during his stay in Toruń where a significant part of this work was done. We thank the referee of the paper for several useful comments and especially for his remark about the possibility to drop the assumption of unique ergodicity of S in the main result of the paper.

1. Notation. Preliminaries

NONSINGULAR TRANSFORMATIONS AND GROUP ACTIONS. Let $(X, \mathfrak{B}_X, \mu)$ be a standard probability space. The group of μ-nonsingular transformations of X will be denoted by $\mathrm{Aut}(X, \mu)$. There exists a natural embedding $T \mapsto U_T$ of $\mathrm{Aut}(X, \mu)$ into the unitary group of $L^2(X, \mu)$ given by

$$U_T f(x) = f(T^{-1}x)\sqrt{\frac{d\mu \circ T^{-1}}{d\mu}(x)}, \quad f \in L^2(X, \mu),\ x \in X.$$

Then $\mathrm{Aut}(X, \mu)$ endowed with the weak operator topology is a Polish group. The subgroup $\mathrm{Aut}_0(X, \mu)$ of μ-preserving transformations is closed in $\mathrm{Aut}(X, \mu)$.

Let G be a l.c.s.c. group. An ergodic nonsingular action $S = (S_g)_{g \in G}$ of G on $(X, \mathfrak{B}_X, \mu)$ is called **of type** I if μ is supported by a single orbit of S. Otherwise S is called **properly ergodic**. Given two nonsingular G-actions $S = (S_g)_{g \in G}$ and $Q = (Q_g)_{g \in G}$ on $(X, \mathfrak{B}_X, \mu)$ and $(Y, \mathfrak{B}_Y, \nu)$ respectively, we denote by $S \times Q$ (resp. $S \otimes Q$) the following G- (resp. $G \times G$-) action on the product space $(X \times Y, \mathfrak{B}_X \otimes \mathfrak{B}_Y, \mu \times \nu)$:

$$(S \times Q)(g) = S(g) \times Q(g), \quad (S \otimes Q)(g, h) = S(g) \times Q(h), \quad g, h \in G.$$

A properly ergodic action S is called **mildly mixing** (see [FuW], [SWa]) if for any properly ergodic G-action Q, the action $S \times Q$ is ergodic. As was shown in

[SWa], such an S preserves an equivalent invariant probability measure. Moreover, a probability preserving S is mildly mixing if and only if for any sequence $g_n \to \infty$ in G and a measurable subset $A \in \mathfrak{B}_X$ with $\lim_{n\to\infty} \mu(S_{g_n} A \triangle A) = 0$, we have $\mu(A) = 0$ or $\mu(A) = 1$. Hence for any noncompact closed subgroup $H \subset G$, the action $S(H)$ is also mildly mixing.

For an action S of G, we denote by $C(S)$ the **centralizer** of S, i.e.,

$$C(S) = \{T \in \mathrm{Aut}(X, \mu) |\, TS_g = S_g T \text{ for all } g \in G\}.$$

For a single transformation T, $C(T)$ denotes $C(\{T^n \mid n \in \mathbb{Z}\})$.

By a **cocycle** of a nonsingular transformation T on $(X, \mathfrak{B}_X, \mu)$ with values in G we mean a measurable map from X to G. The set of all such cocycles is denoted by $Z^1(T, G)$. Endowed with the topology of convergence in measure it is a Polish space. Two cocycles $\phi, \psi \in Z^1(T, G)$ are called **cohomologous** if

$$\phi(x) = a(x)\psi(x)a(Tx)^{-1}$$

for some measurable map $a\colon X \to G$ at a.a. $x \in X$.

Joinings and disjointness. Given two transformations $T_i \in \mathrm{Aut}_0(X_i, \mu_i)$, we denote by $J(T_1, T_2)$ the set of **joinings** of T_1 and T_2, i.e., the set of $T_1 \times T_2$-invariant measures η on $\mathfrak{B}_{X_1} \otimes \mathfrak{B}_{X_2}$ whose marginal on $\mathfrak{B}_{X_i}$ is μ_i, $i = 1, 2$. The corresponding dynamical system $(X_1 \times X_2, \mathfrak{B}_{X_1} \otimes \mathfrak{B}_{X_2}, \eta, T_1 \times T_2)$ is also called a joining of T_1 and T_2. By $J^e(T_1, T_2) \subset J(T_1, T_2)$ we denote the subset of ergodic joinings (it is nonempty whenever T_1 and T_2 are ergodic). Considering three transformations T_1, T_2 and T_3 we define in a similar way $J(T_1, T_2, T_3)$ and $J^e(T_1, T_2, T_3)$. If $J(T_1, T_2) = \{\mu_1 \times \mu_2\}$ then T_1 and T_2 are called **disjoint** [Fu1]. This will be denoted by $T_1 \perp T_2$. If $T_1 = T_2 =: T$ we speak about self-joinings of T and use notation $J_2(T)$ for $J(T_1, T_2)$. Given an extension

$$(X, \mathfrak{B}_X, \mu, T) \to (Y, \mathfrak{B}_Y, \nu, S),$$

consider the desintegration of μ with respect to ν: $\mu = \int_Y \mu_y d\nu(y)$. If now $\eta \in J_2(S)$ then the measure $\widetilde{\eta} := \int_{Y\times Y} \mu_y \times \mu_{y'} d\eta(y, y')$ is a self-joining of T. It is called the **relatively independent extension** of η. Let Δ_Y stand for the diagonal self-joining of S. Assuming that S is ergodic, the extension $T \to S$ is called **relatively weakly mixing** if the relatively independent extension of Δ_Y is ergodic. An ergodic transformation T of $(X, \mathfrak{B}, \mu)$ is called **semisimple** [JLM] if for each $\eta \in J_2^e(T)$, the extension $(T \times T, \eta) \to (T, \mu)$ is relatively weakly mixing. Recall also that T is **2-fold simple** [JRu] if every $\eta \in J_2^e(T)$ is either the product measure $\mu \times \mu$ or a graph joining, i.e., the joining supported by the graph of some $R \in C(T)$.

Given a class $\mathcal{A}$ of ergodic transformations, by $\mathcal{M}(\mathcal{A})$ we denote the class of **multipliers** of $\mathcal{A}$ [Gl1], i.e., the class of transformations whose all ergodic joinings with an arbitrary element of $\mathcal{A}$ give rise to a transformation that is still in $\mathcal{A}$. Let $\mathcal{W}$ and $\mathcal{D}$ stand for the classes of weakly mixing transformations and distal transformations respectively, see [Fu2]. Summarizing the results on disjointness from [Fu1], [GlW], [Gl1] and [LeP] we can write

$$\mathcal{D} \subsetneqq \mathcal{M}(\mathcal{W}^{\perp}) \subsetneqq \mathcal{W}^{\perp}.$$

For a detailed account on joinings and related things we refer to [JRu], [Th] and [Gl2].

Orbit theory and cocycles. We will now briefly recall basics of the orbit theory. The facts we present below can be found in [Sc], [FM], [GS2], [Da2]. The reader should be aware that these facts are not all obvious.

Assume that T is an ergodic nonsingular transformation of $(X, \mathfrak{B}_X, \mu)$. Let $\mathcal{R}$ stand for the T-orbital equivalence relation. We recall definitions of the full group $[\mathcal{R}]$ of $\mathcal{R}$ and its normalizer $N[\mathcal{R}]$:

$$[\mathcal{R}] = \{S \in \operatorname{Aut}(X, \mu) |\ (x, Sx) \in \mathcal{R} \text{ for } \mu\text{-a.a. } x\},$$
$$N[\mathcal{R}] = \{S \in \operatorname{Aut}(X, \mu) |\ S[\mathcal{R}]S^{-1} = [\mathcal{R}]\}.$$

We will also use the notation $[T]$ for $[\mathcal{R}]$. A measurable map $\alpha: \mathcal{R} \to G$ is called a **cocycle** of $\mathcal{R}$ if

$$\alpha(x, y)\alpha(y, z) = \alpha(x, z) \quad \text{for all } (x, y), (y, z) \in \mathcal{R}.$$

Two cocycles $\alpha, \beta: \mathcal{R} \to G$ are said to be **cohomologous** (we then write $\alpha \approx \beta$) if there exists a measurable map $a: X \to G$ such that $\alpha(x, y) = a(x)\beta(x, y)a(y)^{-1}$ for a.a. $(x, y) \in \mathcal{R}$. Two cocycles $\alpha, \beta: \mathcal{R} \to G$ are called **weakly equivalent** if $\alpha \approx \beta \circ \theta$ for some $\theta \in N[\mathcal{R}]$. (The cocycle $\beta \circ \theta$ is defined by $\beta \circ \theta(x, y) = \beta(\theta x, \theta y)$.) Given a cocycle α of $\mathcal{R}$, we set $\phi_\alpha(x) := \alpha(Tx, x)$, $x \in X$. It is easy to check that the map $\alpha \mapsto \phi_\alpha$ is a bijection between the $\mathcal{R}$-cocycles and the T-cocycles. Moreover, $\alpha \approx \beta$ if and only if ϕ_α is cohomologous to ϕ_β.

Recall that λ_G denotes a left Haar measure on G. Let us fix a probability measure λ on G equivalent to λ_G. We define the following nonsingular transformations on $(X \times G, \mathfrak{B}_X \otimes \mathfrak{B}_G, \mu \times \lambda)$:

$$T_\phi(x, g) = (Tx, \phi(x)g), \quad R_h(x, g) = (x, gh^{-1}), \quad h \in G.$$

The cocycle ϕ is called **recurrent** (resp. **ergodic**) if T_ϕ is conservative (resp. ergodic). Notice that $(R_h)_{h\in G}$ is a G-action commuting with T_ϕ. Hence it induces a nonsingular G-action $W_\phi = (W_\phi(g))_{g\in G}$ on the space $(\Omega_\phi, \mathfrak{B}_{\Omega_\phi}, \nu_\phi)$ of T_ϕ-ergodic components. This space is just $(X \times G, \mathfrak{F}, (\mu \times \lambda) \upharpoonright \mathfrak{F})$, where $\mathfrak{F} \subset \mathfrak{B}_X \odot \mathfrak{B}_G$ denotes the σ-algebra of T_ϕ-invariant subsets. W_ϕ is called the **Mackey action** (or the **associated action**) of ϕ. Since T is ergodic, so is W_ϕ.

If there exists a closed subgroup $H \subset G$ such that ϕ is cohomologous to an ergodic cocycle with values in H, then ϕ is called **regular**. The subgroup H turns out to be determined by ϕ up to conjugacy and it is always amenable. Moreover, H is equal to the stabilizer of a point from Ω_ϕ. It can be shown that ϕ is regular if and only if ν_ϕ is supported by a single orbit (i.e., W_ϕ is of type I). Clearly, ϕ is ergodic if and only if W_ϕ is the trivial action on a singleton.

Next, if ϕ corresponds to a cocycle α of $\mathcal{R}$ (i.e., $\phi = \phi_\alpha$) then we will also write W_α for W_ϕ and call α recurrent, regular or ergodic if so is ϕ. Notice that if α is weakly equivalent to β and α is recurrent, regular or ergodic, then so is β. Moreover, if α and β are weakly equivalent then W_α and W_β are isomorphic. A theorem of Golodets and Sinelshchikov states that conversely, if T is measure preserving, α and β are both recurrent with W_α and W_β isomorphic, then α and β are weakly equivalent [GS2].

2. Rokhlin extensions and locally compact group extensions

Let $\widetilde{T}$ be an ergodic measure preserving transformation on a standard probability space $(Z, \mathfrak{B}_Z, \kappa)$ and let $\mathfrak{F} \subset \mathfrak{B}_Z$ be a factor of $\widetilde{T}$. By a classical theorem of Abramov–Rokhlin [AbR], the dynamical system $(Z, \mathfrak{B}_Z, \kappa, \widetilde{T})$ can be represented in a skew product form as follows:

$$(Z, \mathfrak{B}_Z, \kappa) = (X, \mathfrak{B}_X, \mu) \otimes (Y, \mathfrak{B}_Y, \nu) \quad \text{and} \quad \widetilde{T}(x, y) = (Tx, \psi(x)y),$$

where T is an ergodic transformation of $(X, \mathfrak{B}_X, \mu)$ and $\psi\colon X \to \mathrm{Aut}_0(Y, \nu)$ is a measurable map (sometimes called **Rokhlin cocycle** of T). In such a representation $\mathfrak{F}$ corresponds to $\mathfrak{B}_X$ (or, more precisely to $\mathfrak{B}_X \otimes \mathfrak{N}_Y$, where $\mathfrak{N}_Y$ stands for the trivial sub-σ-algebra of $\mathfrak{B}_Y$).

In this paper we mainly study extensions $\widetilde{T} \to T$ of a special form. Namely, let $S = (S_g)_{g\in G}$ be an ergodic measure preserving action of a l.c.s.c. group G on a standard probability space $(Y, \mathfrak{B}_Y, \nu)$. Take $\phi \in Z^1(X, G)$. Then we set

$$\widetilde{T}(x, y) := (Tx, S_{\phi(x)}y)$$

and denote this extension by $T_{\phi,S}$. The case of compact G was deeply investigated by a number of authors (see, e.g., the bibliography in [LeL]). It will not be considered in this paper. In case $G = \mathbb{Z}^n$ or $\mathbb{R}^n$, extensions $T_{\phi,S} \to T$ were studied in [An], [Ki], [Ru], [GlW], [Gl1], [Ro], etc. Later, a more general case of Abelian G was under consideration in [LeL], [LeP], [LNM]. As far as we know, non-Abelian G were not studied in this context (except of some simple facts from [LeL]).

We start with an observation that $T_{\phi,S} \to T$ is not a 'special' extension. In fact, every extension is isomorphic (as an extension) to such a one.

PROPOSITION 2.1: *Let $\widetilde{T} \to T$ be an ergodic extension and let ψ be the corresponding Rokhlin cocycle of T as above. Then there exist a countable amenable group Σ (it does not depend on ψ) which acts ergodically on $(Y, \mathfrak{B}_Y, \nu)$ and a measurable cocycle ϕ: $X \to \Sigma$ such that ψ is cohomologous to ϕ in $\mathrm{Aut}_0(Y, \nu)$ (the natural embedding $\Sigma \subset \mathrm{Aut}_0(Y, \nu)$ is implicit here). Thus $\widetilde{T} \to T$ is isomorphic to $T_{\phi,\Sigma} \to T$.*

Proof: It is easy to see that $\mathrm{Aut}_0(Y, \nu)$ contains a dense countable subgroup Σ which is amenable in the discrete topology. Actually, if ν has an atom then (Y, ν) is measurably isomorphic to a finite cyclic group endowed with Haar measure. Therefore $\mathrm{Aut}_0(Y, \nu)$ is finite and hence amenable. If ν is nonatomic then we can represent (Y, ν) as $\bigotimes_{n=1}^{\infty}(\{0, 1\}, \lambda)$ with $\lambda(0) = \lambda(1) = 0.5$. Let Σ_{2^n} denote the permutation group of $\{0, 1\}^n$. It acts on (Y, ν) permutating the first n coordinates. Then we have $\Sigma_2 \subset \Sigma_4 \subset \cdots \subset \mathrm{Aut}_0(Y, \nu)$. Clearly, the locally finite countable (and hence amenable) group $\Sigma := \bigcup_{n=1}^{\infty} \Sigma_{2^n}$ is dense in $\mathrm{Aut}_0(Y, \nu)$. Hence Σ is an ergodic transformation group. By [Da2, Proposition 1.6], ψ is cohomologous to a cocycle ϕ taking values in Σ. ∎

3. Group self-joinings

A closed subgroup $H \subset G \times G$ is called a **group self-joining** of G if the two coordinate projections of H to G are onto. Put $H_1 := \{g \in G |\ (g, 1_G) \in H\}$ and $H_2 := \{g \in G |\ (1_G, g) \in H\}$. Then H_1 and H_2 are closed normal subgroups of G and, moreover, there exists a topological group isomorphism θ: $G/H_2 \to G/H_1$ such that

$$H = \{(g_1, g_2) \in G^2 |\ \theta(g_2 H_2) = g_1 H_1\}. \tag{3.1}$$

Conversely, given two closed normal subgroups H_1, H_2 of G and a topological group isomorphism θ: $G/H_2 \to G/H_1$, by (3.1), we obtain a group self-joining H of G.

We denote the set of all group self-joinings of G by $J_2(G)$. Given $H \in J_2(G)$, we have a natural topological G^2-action Q_H on G/H_1:

$$Q_H(g_1, g_2)gH_1 = g_1 g H_1 \theta(g_2 H_2)^{-1} \quad \text{for all } g, g_1, g_2 \in G.$$

Clearly, a left Haar measure λ_{G/H_1} is Q_H-quasi-invariant. Slightly abusing notation, we will denote the coordinate G-actions given by the subgroups $G \times \{1_G\}$ and $\{1_G\} \times G$ by $Q(G \times \{1_G\})$ and $Q(\{1_G\} \times G)$ respectively. Notice that these actions are transitive. Now we prove a converse to that.

LEMMA 3.1: *Let Q be a nonsingular action of G^2 on a standard probability space $(Z, \mathfrak{B}_Z, \kappa)$ such that the G-actions $Q(G \times \{1_G\})$ and $Q(\{1_G\} \times G)$ are ergodic and of type I. Then there exists $H \in J_2(G)$ such that Q is isomorphic to Q_H.*

Proof: Denote the G-actions $Q(G \times \{1_G\})$ and $Q(\{1_G\} \times G)$ by Q_1 and Q_2 respectively. Since Q_1 is ergodic and of type I, there exists a closed subgroup $H_1 \subset G$ such that Z is measurably isomorphic to the homogeneous space G/H_1 and Q_1 is the action by left translations; moreover, κ is equivalent to a Haar measure on G/H_1. Denote by $N_G(H_1)$ the normalizer of H_1 in G, i.e.,

$$N_G(H_1) = \{g \in G |\ g^{-1} H_1 g = H_1\}.$$

Then the quotient group $N_G(H_1)/H_1$ acts on $(G/H_1, \kappa)$ by inverted right translations:

$$(nH_1) \cdot (gH_1) = gH_1 n^{-1}, \quad \text{for all } g \in G \text{ and } n \in N_G(H_1).$$

Notice that $C(Q_1) = N_G(H_1)/H_1$ (see, for example, [Da1]). Since $Q_2(G) \subset C(Q_1)$ and Q_2 is ergodic and of type I, we conclude that $N_G(H_1)/H_1$ acts transitively on G/H_1. It is easy to verify that this happens if and only if H_1 is normal in G. Moreover, Q_2 determines an epimorphism θ' of G onto G/H_1 such that

$$Q_2(g) \cdot g' H_1 = g' H_1 \theta'(g)^{-1} \quad \text{for all } g, g' \in G.$$

It remains to set $H_2 := \operatorname{Ker} \theta'$ and $H := \{(g_1, g_2) \in G^2 |\ \theta'(g_2) = g_1 H_1\}$. ∎

4. Mackey actions for $\phi \times \phi \circ R$

Let T be an ergodic measure preserving transformation of $(X, \mathfrak{B}_X, \mu)$ and $\phi, \psi \in Z^1(T, G)$. The associated actions W_ϕ, W_ψ and $W_{\phi \times \psi}$ are connected by the following duality.

LEMMA 4.1:

(i) *W_ϕ is isomorphic to the restriction of $W_{\phi\times\psi}(G \times \{1_G\})$ to the σ-algebra of $W_{\phi\times\psi}(\{1_G\} \times G)$-invariant subsets, and*

(ii) *W_ψ is isomorphic to the restriction of $W_{\phi\times\psi}(\{1_G\} \times G)$ to the σ-algebra of $W_{\phi\times\psi}(G \times \{1_G\})$-invariant subsets.*

Proof: We only need to demonstrate (i). Let $\mathfrak{F}_\phi \subset \mathfrak{B}_X \otimes \mathfrak{B}_G$ and $\mathfrak{F}_{\phi\otimes\psi} \subset \mathfrak{B}_X \otimes \mathfrak{B}_G \otimes \mathfrak{B}_G$ stand for the σ-algebras of T_ϕ- and $T_{\phi\times\psi}$-invariant subsets respectively. Consider the sub-σ-algebra $\mathfrak{S}$ of those subsets $A \in \mathfrak{F}_{\phi\times\psi}$ which are invariant under all translations along the 'third' coordinate. It is easy to see that $A = A' \times G$ for a subset $A' \in \mathfrak{B}_X \otimes \mathfrak{B}_G$. Since $A \in \mathfrak{F}_{\phi\times\psi}$, it follows that $A' \in \mathfrak{F}_\phi$. Thus we obtain a Boolean isomorphism $\mathfrak{F}_\phi \ni A' \mapsto A' \times G \in \mathfrak{S}$ intertwining $W_\phi(g)$ with $W_{\phi\times\psi}(g, 1_G)$ for all $g \in G$. ∎

By an immediate use of the lemma we get the following.

PROPOSITION 4.2: *If ϕ is ergodic and $R \in C(T)$, then the coordinate G-actions $W_{\phi\times\phi\circ R}(G \times \{1_G\})$ and $W_{\phi\times\phi\circ R}(\{1_G\} \times G)$ are both ergodic.*

We intend to prove a converse to Proposition 4.2 under an additional assumption that $R^n \notin [T]$ for all $n \neq 0$. It is easy to check that this is equivalent to the following: $R^n \neq T^m$ for all $n, m \in \mathbb{Z}$ with $n^2 + m^2 \neq 0$. In turn, this means that the joint $\mathbb{Z}^2$-action generated by R and T is free.

PROPOSITION 4.3: *Let G be amenable and let V be a nonsingular ergodic G^2-action. Suppose that the G-actions $V(G \times \{1_G\})$ and $V(\{1_G\} \times G)$ are both ergodic. Then under the above assumption on R there exists an ergodic T-cocycle $\phi: X \to G$ such that V is conjugate to the G^2-action associated to the product T-cocycle $\phi \times \phi \circ R$.*

Proof: It is convenient to make use of the language of the orbit theory in the proof. Let $\mathcal{R}$ stand for the T-orbit equivalence relation. By a theorem of Golodets–Sinelshchikov [GS1], there exists a recurrent cocycle

$$\alpha = \alpha_1 \times \alpha_2: \mathcal{R} \to G \times G$$

such that the associated action W_α is conjugate to V. By Lemma 4.1, the Mackey G-action W_{α_1} is just the restriction of $W_\alpha(G \times \{1_G\})$ to the σ-algebra of $W_\alpha(\{1_G\} \times G)$-invariant subsets. However, this σ-algebra is trivial since $V(\{1_G\} \times G)$ is ergodic. Thus W_{α_1} is the trivial action on a singleton. Hence α_1 is ergodic. In a similar way, α_2 is ergodic as well. Then by the uniqueness theorem for ergodic cocycles [GS2], there exists a transformation $Q \in N[\mathcal{R}]$ such that the cocycles $\alpha_1 \circ Q$ and α_2 are cohomologous. By a standard trick in the orbit theory (see [GS2], [Da1]) replacing, if necessary, α by a weakly equivalent cocycle we can assume without loss of generality that $Q^n \notin [\mathcal{R}]$ for all nonzero $n \in \mathbb{Z}$, i.e., Q is outer aperiodic in the sense of [CK]. On the other hand, by the assumptions, R is also outer aperiodic. Then the Connes–Krieger outer conjugacy theorem [CK] implies that $Q = tLRL^{-1}$ for some transformations $t \in [\mathcal{R}]$ and $L \in N[\mathcal{R}]$. Now we have

$$\begin{aligned}\alpha &= \alpha_1 \times \alpha_2 \approx \alpha_1 \times \alpha_1 \circ Q = \alpha_1 \times \alpha_1 \circ t \circ (LRL^{-1}) \approx \alpha_1 \times \alpha_1 \circ (LRL^{-1}) \\ &= (\alpha_1 \circ L \times \alpha_1 \circ L \circ R) \circ L^{-1}.\end{aligned}$$

Denote the cocycle $\alpha_1 \circ L$ by β. Then α is weakly equivalent to $\beta \times \beta \circ R$. Since the isomorphism class of the associated Mackey action is invariant under the weak equivalence of the underlying cocycles, the action $W_{\beta\times\beta\circ R}$ of G^2 is conjugate to V. It remains to define ϕ: $X \to G$ by setting $\phi(x) := \beta(x, Tx)$ and notice that

$$\beta \circ R(x, Tx) = \phi(Rx) \quad \text{for a.a. } x \in X. \qquad \blacksquare$$

Remark 4.4: Using the same argument one can extend Proposition 4.3 as follows. Let V be a nonsingular ergodic G^2-action. Then there exists a recurrent T-cocycle ϕ: $X \to G$ such that $W_{\phi\times\phi\circ R}$ is conjugate to V if and only if the restriction of $V(G \times \{1_G\})$ to the σ-algebra of $V(\{1_G\} \times G)$-invariant subsets is isomorphic to the restriction of $V(\{1_G\} \times G)$ to the σ-algebra of $V(G \times \{1_G\})$-invariant subsets.

5. Ergodic decomposition and r.f.m.p. factors

Let $S = (S_g)_{g\in G}$ be a Borel action of a l.c.s.c. group G on a standard Borel space $(Y, \mathfrak{B}_Y)$. Let α: $G \times Y \to \mathbb{R}_+^*$ be a Borel map satisfying the following cocycle identity,

$$\alpha(g_2 g_1, y) = \alpha(g_2, S_{g_1} y)\alpha(g_1, y) \quad \text{for all } y \in Y \text{ and } g_1, g_2 \in G.$$

Denote by $\mathcal{P}$ the set of S-quasi-invariant probability measures on $(Y, \mathfrak{B}_Y)$. Given $\nu \in \mathcal{P}$, we set

$$\mathcal{P}_\alpha := \left\{\lambda \in \mathcal{P} \,\middle|\, \frac{d\lambda \circ S_g}{d\lambda}(y) = \alpha(g, y) \text{ at } \lambda\text{-a.e. } y \text{ for every } g \in G\right\} \quad \text{and}$$

$$\mathcal{E}_\alpha := \{\lambda \in \mathcal{P}_\alpha \,|\, S \text{ is ergodic with respect to } \lambda\}.$$

Notice that $\mathcal{P}_\alpha$ can be empty. Suppose this is not the case. Then clearly, $\mathcal{P}_\alpha$ is convex and $\mathcal{E}_\alpha$ is the set of extremal points of $\mathcal{P}_\alpha$. Notice that $\mathcal{P}_\alpha$ furnished with the natural Borel σ-algebra $\mathfrak{B}_{\mathcal{P}_\alpha}$ (making the map $\mathcal{P}_\alpha \ni \lambda \mapsto \lambda(B) \in \mathbb{R}$ Borel for any $B \in \mathfrak{B}_Y$) is a standard Borel space and $\mathcal{E}_\alpha$ is a Borel subset of it [GrS]. In view of the following lemma, $\mathcal{P}_\alpha$ can be interpreted as a Borel 'simplex' of nonsingular measures.

LEMMA 5.1 ([GrS]): *Given $\nu \in \mathcal{P}$, fix a Borel variant α_ν: $G \times Y \to \mathbb{R}_+^*$ of the Radon–Nikodym derivative of (S, ν). Then there exists a unique probability measure κ on $\mathcal{E}_{\alpha_\nu}$ such that*

$$\nu = \int_{\mathcal{E}_{\alpha_\nu}} \epsilon d\kappa(\epsilon). \tag{5.1}$$

Moreover, if $\mathfrak{F}$ stands for the σ-algebra of S-invariant subsets then $(\mathcal{E}_{\alpha_\nu}, \mathfrak{B}_{\mathcal{E}_{\alpha_\nu}} \kappa)$ is identified naturally with $(Y, \mathfrak{F}, \nu \restriction \mathfrak{F})$.

For a measure $\nu \in \mathcal{P}$, let $\mathfrak{F}$ be a factor of $(Y, \mathfrak{B}_Y, \nu, S)$. If S preserves ν and S is ergodic on $\mathfrak{F}$, then $\epsilon \restriction \mathfrak{F} = \nu \restriction \mathfrak{F}$ for κ-a.e. ϵ in (5.1). This 'good projection' property no longer holds for an arbitrary S-quasi-invariant measure ν. However, we will show that it survives in an important special 'nonsingular' case.

Definition 5.2: Given a measure $\nu \in \mathcal{P}$, a factor $\mathfrak{F}$ (and the extension $S \to S \restriction \mathfrak{F}$) is called **relatively finite measure preserving** (r.f.m.p.) if the Radon–Nikodym derivative $d\nu \circ S_g / d\nu$ is $\mathfrak{F}$-measurable for all $g \in G$.

In particular, $S \to S \restriction \mathfrak{N}_Y$ is r.f.m.p. if and only if S preserves ν. (Recall that $\mathfrak{N}_Y$ stands for the trivial sub-σ-algebra of $\mathfrak{B}_Y$.) Moreover, it is easy to verify that if $S \to S \restriction \mathfrak{F}$ is r.f.m.p. and $S \restriction \mathfrak{F}$ admits an equivalent invariant (finite or σ-finite) measure, then so does S (it also follows from (5.2) below).

We can restate Definition 5.2 in an equivalent way. Denote the dynamical system $(Y, \mathfrak{F}, \nu \restriction \mathfrak{F}, S \restriction \mathfrak{F})$ by $(Z, \mathfrak{B}_Z, \kappa, V)$. Let π: $Y \to Z$ stand for the corresponding projection and $\nu = \int_Z \nu_z d\kappa(z)$ be the desintegration of ν with respect to κ. Then

$$\frac{d\nu \circ S_g}{d\nu}(y) = \frac{d\kappa \circ V(g)}{d\kappa}(\pi(y)) \frac{d\nu_{V(g)\pi(y)} \circ S_g}{d\nu_{\pi(y)}}(y)$$

at ν-a.e. y for all $g \in G$. Hence $\mathfrak{F}$ is r.f.m.p. if and only if

$$\frac{d\nu_{V(g)\pi(y)} \circ S_g}{d\nu_{\pi(y)}}(y) = 1 \quad \text{at } \nu\text{-a.e. } y \text{ for all } g \in G, \quad \text{i.e.}$$

$$\nu_{V(g)\pi(y)} \circ S_g = \nu_{\pi(y)} \quad \text{for all } g \in G. \tag{5.2}$$

Now we see that if $\mathfrak{F}$ is r.f.m.p. (with respect to ν), then by Lemma 5.1 the Radon–Nikodym derivative $d\epsilon \circ S_g/d\epsilon$ is $\mathfrak{F}$-measurable for κ-a.e. ϵ in (5.1) and all $g \in G$. Suppose in addition that $S \restriction \mathfrak{F}$ is ergodic. Since $\nu \restriction \mathfrak{F} = \int_{\mathcal{E}_{\alpha_\nu}} \epsilon \restriction \mathfrak{F} d\kappa(\epsilon)$, it follows from the uniqueness part of Lemma 5.1 that $\epsilon \restriction \mathfrak{F} = \nu \restriction \mathfrak{F}$ for κ-a.e. ϵ. Thus we have proved the following.

PROPOSITION 5.3: *Let $\nu \in \mathcal{P}$. If $\mathfrak{F}$ is an ergodic r.f.m.p. factor of $(Y, \mathfrak{B}_Y, \nu, S)$, then for κ-a.e. ϵ from (5.1), the restriction of ϵ to $\mathfrak{F}$ is equal to $\nu \restriction \mathfrak{F}$.*

We will also need the following simple lemma about r.f.m.p. extensions.

LEMMA 5.4: *Let S be an ergodic nonsingular G-action on a standard probability space $(Y, \mathfrak{B}_Y, \nu)$ and let ρ be an $(S \times \mathrm{Id})$-quasi-invariant measure on $(Y \times Z, \mathfrak{B}_Y \otimes \mathfrak{B}_Z)$. Assume that $(Y \times Z, \mathfrak{B}_Y \otimes \mathfrak{B}_Z, \rho, S \times \mathrm{Id}) \to (Y, \mathfrak{B}_Y, \nu, S)$ is an r.f.m.p. extension. Then $\rho = \nu \times \kappa$ for a probability measure κ on $\mathfrak{B}_Z$.*

Proof: Passing, if necessary, to a dense countable subgroup we may assume without loss of generality that G is countable. Let $\rho = \int (\delta_y \times \rho_y) d\nu(y)$ be the desintegration of ρ with respect to ν. It follows from (5.2) that $\rho_{S_g y} = \rho_y$ a.e. in ν for all $g \in G$. Since S is ergodic and the map $Y \ni y \mapsto \rho_y$ is measurable, the result follows. ∎

Now we give a natural example of r.f.m.p. factors.

We will need the following nonsingular version of the Abramov–Rokhlin theorem on factors (see [Ra]):

Let V be an ergodic nonsingular action of G on a standard probability space $(Z, \mathfrak{B}_Z, \kappa)$ and let $\mathfrak{F}$ be a factor of V (i.e., a V-invariant sub-σ-algebra). Then there exist a measure space isomorphism Λ of $(Z, \mathfrak{B}_Z, \kappa)$ onto a product measure space $(X, \mathfrak{B}_X, \mu) \times (Y, \mathfrak{B}_Y, \nu)$, a nonsingular action W of G on $(X, \mathfrak{B}_X, \mu)$ and a Borel cocycle

$$F\colon G \times X \ni (g, x) \mapsto F(g, x) \in \mathrm{Aut}(Y, \nu)$$

such that $\{\Lambda(F) |\, F \in \mathfrak{F}\} = \{B \times Y |\, B \in \mathfrak{B}_X\} \pmod 0$ and

$$\Lambda V(g) \Lambda^{-1}(x, y) = (W(g)x, F(g, x)y)$$

at a.a. (x, y) for all $g \in G$.

PROPOSITION 5.5: *Let V be an ergodic nonsingular action of G on a standard probability space $(Z, \mathfrak{B}_Z, \kappa)$ and let R be a κ-preserving transformation from* *the centralizer $C(V)$. Then the σ-algebra $\mathfrak{F}$ of R-invariant sets is an r.f.m.p. factor of V.*

Proof: By the nonsingular version of the Abramov–Rokhlin theorem, we may assume

$$(Z, \mathfrak{B}_Z, \kappa) = (X, \mathfrak{B}_X, \mu) \otimes (Y, \mathfrak{B}_Y, \nu),$$
$$V(g)(x, y) = (W(g)x, F(g, x)y), \ g \in G,$$

where $\mathfrak{F} = \mathfrak{B}_X \otimes \{\emptyset, Y\}$, W is the restriction of V to $\mathfrak{F}$ and

$$F: G \times X \ni (g, x) \mapsto F(g, x) \in \mathrm{Aut}(Y, \nu)$$

is a Borel cocycle of W. Since $\mathfrak{F}$ is a factor of R as well and R acts as the identity on $\mathfrak{F}$, it follows that $R(x, y) = (x, R_x y)$ at a.a. (x, y) for a measurable field of nonsingular transformations $X \ni x \mapsto R_x \in \mathrm{Aut}(Y, \nu)$. Moreover, these transformations R_x are ergodic for a.a. x as the extension $\mathfrak{B}_Z \to \mathfrak{F}$ yields the R-ergodic decomposition. Since R preserves $\mu \times \nu$, we conclude immediately that R_x preserves ν for μ-a.e. x. Moreover, since

$$R^{-1}V(g)R(x, y) = (W(g)x, R^{-1}_{W(g)x} F(g, x) R_x y) = V(g)(x, y),$$

it follows that $R^{-1}_{W(g)x} F(g, x) R_x = F(g, x)$ at a.a. x for all $g \in G$. Hence

$$\frac{d\nu \circ F(g, x)}{d\nu}(y) = \frac{d\nu \circ F(g, x)}{d\nu}(R_x y)$$

at ν-a.e. y for μ-a.a. x and all $g \in G$. Therefore $d\nu \circ F(g, x)/d\nu$ is a constant ν-a.e. and this constant is obviously equal to 1, i.e., $F(g, x)$ preserves ν for μ-a.e. x and all $g \in G$. The latter is equivalent to the r.f.m.p. property of $\mathfrak{F}$ by (5.2). ∎

Notice that the above proposition is a natural generalization of the well-known fact that a nonsingular transformation commuting with an ergodic probability preserving transformation is itself measure preserving.

The proposition below will be used in the proof of the main result of the paper. Let T be an ergodic nonsingular transformation of $(X, \mathfrak{B}_X, \mu)$ and R a measure preserving transformation of $(Z, \mathfrak{B}_Z, \kappa)$ such that $T \times R$ is ergodic. Let $\phi \in Z^1(T, G)$. By $\phi \otimes 1$ we denote the following cocycle of $T \times R$:

$$\phi \otimes 1(x, z) = \phi(x), \quad (x, z) \in X \times Z.$$

Recall that a probability measure λ equivalent to a left Haar measure on G is fixed and that $(\Omega_\phi, \mathfrak{B}_{\Omega_\phi}, \nu_\phi)$ stands for the space of the Mackey G-action W_ϕ.

Notice that since $T \times R$ is ergodic, the Mackey action $W_{\phi\otimes 1}$ is well defined on its measure space $(\Omega_{\phi\otimes 1}, \mathfrak{B}_{\Omega_{\phi\otimes 1}}, \nu_{\phi\otimes 1})$.

PROPOSITION 5.6: *Assume that T, R and ϕ are as above. Denote by R' the restriction of the transformation* $\mathrm{Id} \times R \times \mathrm{Id} \in \mathrm{Aut}_0(X \times Z \times G, \mu \times \kappa \times \lambda)$ *to the σ-algebra of $(T \times R)_{\phi\otimes 1}$-invariant subsets. Then*

(i) *$R' \in C(W_{\phi\otimes 1})$ and it is a conservative transformation of $(\Omega_{\phi\otimes 1}, \nu_{\phi\otimes 1})$,*
(ii) *the natural projection $\pi\colon (\Omega_{\phi\otimes 1}, \nu_{\phi\otimes 1}) \to (\Omega_\phi, \nu_\phi)$ intertwining $W_{\phi\otimes 1}$ with W_ϕ yields the R'-ergodic decomposition.*

Proof: (i) The transformation $\mathrm{Id} \times R \times \mathrm{Id}$ is conservative since it preserves a finite measure. Hence R' is conservative (as a factor of a conservative map). Clearly, it commutes with $W_{\phi\otimes 1}$.

(ii) It suffices to notice that any R'-invariant subset A' is of the form

$$\{(x, z, g) |\, (x, g) \in A, z \in Z\}$$

for some subset $A \subset X \times G$. Clearly, A' is $(T \times R)_{\phi\otimes 1}$-invariant if and only if A is T_ϕ-invariant. ∎

We deduce from Propositions 5.6 and 5.5 the following.

COROLLARY 5.7: *Under the assumptions of Proposition 5.6, the natural projection π is r.f.m.p.*

Using Corollary 5.7 and the remark just after Definition 5.2 we obtain the following.

COROLLARY 5.8: *Under the assumptions of Proposition 5.6:*

(i) *If W_ϕ admits an equivalent invariant finite (or σ-finite) measure then so does $W_{\phi\otimes 1}$.*
(ii) *If ϕ is ergodic (and hence W_ϕ is trivial) then $W_{\phi\otimes 1}$ preserves $\nu_{\phi\otimes 1}$.*

We note that the assertion (ii) of Corollary 5.8 was established in [LeP] for finite measure preserving T and Abelian G.

6. R.f.m.p. extensions $T_{\phi,S} \to T$ and associated Mackey actions

Let S be a Borel action of G on a standard Borel space $(Y, \mathfrak{B}_Y)$. For an invariant sub-σ-algebra $\mathfrak{F} \subset \mathfrak{B}_Y$ and a quasi-invariant measure κ on $\mathfrak{F}$ we let

$$\mathcal{P}(S, \mathfrak{F}, \kappa) := \{\nu \in \mathcal{P} |\, \nu \restriction \mathfrak{F} = \kappa \text{ and } \mathfrak{F} \text{ is an r.f.m.p. factor of } (Y, \mathfrak{B}_Y, \nu, S)\}.$$

Given an ergodic nonsingular transformation T of $(X, \mathfrak{B}_X, \mu)$ and a cocycle ϕ: $X \to G$ of T, we are interested in the simplex $\mathcal{P}(T_{\phi,S}, \mathfrak{B}_X, \mu)$. Let $R = (R_g)_{g \in G}$ denote the nonsingular G-action on $(G, \mathfrak{B}_G, \lambda)$ by inverted right translations.

Our next statement is a slight modification and extension of a part of Proposition 2.1 from [LeP], where G was assumed Abelian and T measure preserving.

Consider the G-action $\mathrm{Id} \times R \times S$ on the product space

$$(X \times G \times Y, \mathfrak{B}_X \otimes \mathfrak{B}_G \otimes \mathfrak{B}_Y).$$

It obviously commutes with the transformation $T_\phi \times \mathrm{Id}$. Hence their 'joint' $(\mathbb{Z} \times G)$-action, say V, is well defined on $X \times G \times Y$.

Proposition 6.1: *The simplices $\mathcal{P}(V, \mathfrak{B}_X \otimes \mathfrak{B}_G, \mu \times \lambda)$, $\mathcal{P}(T_{\phi,S}, \mathfrak{B}_X, \mu)$ and $\mathcal{P}(W_\phi \times S, \mathfrak{B}_{\Omega_\phi}, \nu_\phi)$ are pairwise affine isomorphic. Moreover, if Λ stands for the corresponding affine isomorphism of $\mathcal{P}(T_{\phi,S}, \mathfrak{B}_X, \mu)$ onto $\mathcal{P}(W_\phi \times S, \mathfrak{B}_{\Omega_\phi}, \nu_\phi)$, then $\Lambda(\mu \times \nu) = \nu_\phi \times \nu$ for any S-invariant measure ν on Y.*

Proof: Take any probability measure η on $X \times G \times Y$ projecting onto $\mu \times \lambda$ and let $\eta = \int_{X \times G} \delta_{(x,g)} \times \eta_{(x,g)} d\mu(x) d\lambda(g)$ be its desintegration. By definition, $\eta \in \mathcal{P}(V, \mathfrak{B}_X \otimes \mathfrak{B}_G, \mu \times \lambda)$ if and only if η is V-quasi-invariant and the extensions

$$(X \times G \times Y, \eta, (\mathrm{Id} \times R_g \times S_g)_{g \in G}) \to (X \times G, \mu \times \lambda, (\mathrm{Id} \times R_g)_{g \in G}),$$
$$(X \times G \times Y, \eta, T_\phi \times \mathrm{Id}) \to (X \times G, \mu \times \lambda, T_\phi)$$

are r.f.m.p. By (5.2) this is equivalent to the following two equations on $\eta_{(x,g)}$:

$$\eta_{(x,gh^{-1})} = \eta_{(x,g)} \circ S_h^{-1}, \tag{6.1}$$

$$\eta_{T_\phi(x,g)} = \eta_{(x,g)} \tag{6.2}$$

at a.e. (x, g) for every $h \in G$. It is a standard fact that the first equation admits a unique solution of the form $\eta_{(x,g)} = \eta_x^* \circ S_g$ at a.a. (x, g) for a measurable field $X \ni x \mapsto \eta_x^*$ of probability measures on Y. The second equation now means that $\eta_{Tx}^* = \eta_x^* \circ S_{\phi(x)}^{-1}$. We define a measure η^* on $X \times Y$ by setting $\eta^* := \int_X \delta_x \times \eta_x^* d\mu(x)$. By (5.2), $\eta^* \in \mathcal{P}(T_{\phi,S}, \mathfrak{B}_X, \mu)$. Clearly, the map $\eta \mapsto \eta^*$ is an affine isomorphism of $\mathcal{P}(V, \mathfrak{B}_X \otimes \mathfrak{B}_G, \mu \times \lambda)$ onto $\mathcal{P}(T_{\phi,S}, \mathfrak{B}_X, \mu)$.

Consider the T_ϕ-ergodic decomposition of $\mu \times \lambda$ (see Lemma 5.1): $\mu \times \lambda = \int_{\Omega_\phi} \omega d\nu_\phi(\omega)$. Then for any

$$\eta = \int_{X \times G} \delta_{(x,g)} \times \eta_{(x,g)} d\mu(x) d\lambda(g) \in \mathcal{P}(V, \mathfrak{B}_X \otimes \mathfrak{B}_G, \mu \times \lambda),$$

we have

$$\eta = \int_{\Omega_\phi} \int_{X\times G} \delta_{(x,g)} \times \eta_{(x,g)} d\omega(x,g) d\nu_\phi(\omega)$$

with $\eta_{(x,g)}$ satisfying (6.1) and (6.2). It follows from (6.2) that $\eta_{(x,g)} = \eta^{\#}_{\omega}$ at ω-a.a. (x,g) for a probability measure $\eta^{\#}_{\omega}$ on Y and ν_ϕ-a.a. ω. Now (6.1) implies that $\eta^{\#}_{W_\phi(g)\omega} = \eta^{\#}_{\omega} \circ S_g^{-1}$ at a.e. ω for all $g \in G$. Let $\eta^{\#}$ be a probability measure on $\Omega_\phi \times Y$ given by $\eta^{\#} = \int_{\Omega_\phi} \delta_\omega \times \eta^{\#}_{\omega} d\nu_\phi(\omega)$. It follows from the construction and (5.2) that the map $\eta \mapsto \eta^{\#}$ is an affine isomorphism of $\mathcal{P}(V, \mathfrak{B}_X \otimes \mathfrak{B}_G, \mu \times \lambda)$ onto $\mathcal{P}(W_\phi \times S, \mathfrak{B}_{\Omega_\phi}, \nu_\phi)$.

The second claim of the proposition can be verified now by a straightforward calculation. ∎

Remark 6.2: Let $\mathfrak{L} \subset \mathfrak{B}_Y$ be an S-invariant sub-σ-algebra. Suppose that for some $\rho \in \mathcal{P}(T_{\phi,S}, \mathfrak{B}_X, \mu)$, we have $\rho \restriction (\mathfrak{B}_X \otimes \mathfrak{L}) = \mu \times \nu_1$, where ν_1 is an S-invariant probability on $(Y, \mathfrak{L})$. Then by the proof of the second claim of Proposition 6.1, $\Lambda(\rho) \restriction (\mathfrak{B}_{\Omega_\phi} \otimes \mathfrak{L}) = \nu_\phi \times \nu_1$.

Remark 6.3 (on functorial properties of $*$ and $\#$)*:* Let A be a measure preserving transformation of a standard probability space $(Z, \mathfrak{B}_Z, \kappa)$ such that the product $T \times A$ is ergodic. Then the map

$$\phi \otimes 1 \colon X \times Z \ni (x,z) \mapsto \phi(x) \in G$$

is a cocycle of $T \times A$. Next, we can define a $\mathbb{Z} \times G$-action V' on

$$(X \times Z \times G \times Y, \mu \times \kappa \times \lambda \times \nu)$$

in a perfect analogy with V. Since A preserves κ, the natural restrictions of measures induce the following affine onto maps:

$$\pi_1 \colon \mathcal{P}(V', \mathfrak{B}_X \otimes \mathfrak{B}_Z \otimes \mathfrak{B}_G, \mu \times \kappa \times \lambda) \to \mathcal{P}(V, \mathfrak{B}_X \otimes \mathfrak{B}_G, \mu \times \lambda),$$
$$\pi_2 \colon \mathcal{P}((T \times A)_{\phi\otimes 1,S}, \mathfrak{B}_X \otimes \mathfrak{B}_Z, \mu \times \kappa) \to \mathcal{P}(T_{\phi,S}, \mathfrak{B}_X, \mu) \quad \text{and}$$
$$\pi_3 \colon \mathcal{P}(W_{\phi\otimes 1} \times S, \mathfrak{B}_{\Omega_{\phi\otimes 1}}, \nu_{\phi\otimes 1}) \to \mathcal{P}(W_\phi \times S, \mathfrak{B}_{\Omega_\phi}, \nu_\phi).$$

We claim that they respect the maps $*$ and $\#$ constructed in the proof of Proposition 6.1, i.e., $\pi_1(\eta)^* = \pi_2(\eta^*)$ and $\pi_1(\eta)^{\#} = \pi_3(\eta^{\#})$ for all

$$\eta \in \mathcal{P}(V', \mathfrak{B}_X \otimes \mathfrak{B}_Z \otimes \mathfrak{B}_G, \mu \times \kappa \times \lambda).$$

We only briefly prove the second formula (the first one is easier and we leave its verification to the reader). Take any $\eta \in \mathcal{P}(V', \mathfrak{B}_{X\times Z\times G}, \mu \times \kappa \times \lambda)$. Then

$$\eta = \int_{\Omega_{\phi\otimes 1}} \omega' \times \eta^{\#}_{\omega'} d\nu_{\phi\otimes 1}(\omega'). \tag{6.3}$$

Next, desintegrate $\nu_{\phi\otimes 1}$ with respect to ν_ϕ as follows:

$$\nu_{\phi\otimes 1} = \int_{\tau^{-1}(\omega)} \xi_\omega d\nu_\phi(\omega), \tag{6.4}$$

where τ: $(\Omega_{\phi\otimes 1}, \nu_{\phi\otimes 1}) \to (\Omega_\phi, \nu_\phi)$ is the natural projection intertwining $W_{\phi\otimes 1}$ with W_ϕ and substitute this into (6.3). By the uniqueness of desintegration, we obtain

$$\int_{\tau^{-1}(\omega)} \eta^{\#}_{\omega'} d\xi_\omega(\omega') = \pi_1(\eta)^{\#}_\omega \quad \text{for a.a. } \omega \in \Omega_\phi.$$

In a similar way, substituting (6.4) into

$$\eta^{\#} = \int_{\Omega_{\phi\otimes 1}} \delta_{\omega'} \times \eta^{\#}_{\omega'} d\nu_{\phi\otimes 1}(\omega')$$

we deduce that

$$\int_{\tau^{-1}(\omega)} \eta^{\#}_{\omega'} d\xi_\omega(\omega') = \pi_3(\eta^{\#})_\omega \quad \text{for a.a. } \omega \in \Omega_\phi.$$

Hence $\pi_1(\eta)^{\#}_\omega = \pi_3(\eta^{\#})_\omega$ for a.a. ω and we are done.

7. Lifting of joinings

We recall that the definitions of $J_2(G)$ and Q_H for an element $H \in J_2(G)$ were given in Section 3. We also notice that an $(S \otimes S)(H)$-invariant measure is both $S(H_1) \otimes \mathrm{Id}$- and $\mathrm{Id} \otimes S(H_2)$-invariant. In order to prove the main result of this section—Theorem 7.3—we need two auxiliary lemmas.

LEMMA 7.1: *Let S_i be an ergodic measure preserving G-action on $(Y_i, \mathfrak{B}_{Y_i}, \nu_i)$, $i = 1, 2$. Assume that Q is a nonsingular G^2-action on a standard probability space $(Z, \mathfrak{B}_Z, \kappa)$ such that the coordinate G-actions $Q(\{1_G\} \times G)$ and $Q(G \times \{1_G\})$ are both ergodic.*

(i) *If S_2 is mildly mixing and $Q(\{1_G\} \times G)$ is properly ergodic, then*

$$\{\rho \in \mathcal{P}((S_1 \otimes S_2) \times Q, \mathfrak{B}_Z, \kappa) | \ \rho \restriction (\mathfrak{B}_{Y_2} \otimes \mathfrak{B}_Z) = \nu_2 \times \kappa \\ \text{and } \rho \restriction \mathfrak{B}_{Y_1} = \nu_1\} = \{\nu_1 \times \nu_2 \times \kappa\}.$$

(ii) *If $Q(\{1_G\} \times G)$ and $Q(G \times \{1_G\})$ are both of type I, then*

$$\rho \in \mathcal{P}((S_1 \otimes S_2) \times Q, \mathfrak{B}_Z, \kappa)$$

if and only if there exist $H \in J_2(G)$ and an $(S \otimes S)(H)$-invariant measure ρ^* on $Y_1 \times Y_2$ such that (up to isomorphism) $Q = Q_H$, $Z = G/H_1$, κ is *equivalent to a left Haar measure λ_{G/H_1} and*

$$\rho = \int_Z \rho^* \circ (S_1(g) \times \mathrm{Id}) \times \delta_{gH_1} d\kappa(gH_1)$$

is the desintegration of ρ relative to κ.

Proof: (i) Take $\rho \in \mathcal{P}((S_1 \otimes S_2) \times Q, \mathfrak{B}_Z, \kappa)$. Then

$$\frac{d\rho \circ (S_1(g_1) \times S_2(g_2) \times Q(g_1, g_2))}{d\rho}(y_1, y_2, z) = \frac{d\kappa \circ Q(g_1, g_2)}{d\kappa}(z) \tag{7.1}$$

for ρ-a.e. (y_1, y_2, z), and all $(g_1, g_2) \in G^2$. Assume additionally that

$$\rho \restriction (\mathfrak{B}_{Y_2} \otimes \mathfrak{B}_Z) = \nu_2 \times \kappa.$$

It follows that the G-action $((S_2(g) \times Q(1_G, g))_{g \in G}, \rho \restriction (\mathfrak{B}_{Y_2} \otimes \mathfrak{B}_Z))$ is ergodic since S_2 is mildly mixing while $Q(\{1_G\} \times G)$ is properly ergodic. Now put $g_1 = 1_G$ in (7.1) and apply Lemma 5.4 to deduce that $\rho = \nu' \times (\nu_2 \times \kappa)$ for a measure ν' on $\mathfrak{B}_{Y_1}$. If we assume in addition that $\rho \restriction \mathfrak{B}_{Y_1} = \nu_1$, then $\nu' = \nu_1$ and (i) follows.

(ii) By Lemma 3.1, there exists $H \in J_2(G)$ such that (up to isomorphism) $Z = G/H_1$, $Q = Q_H$ and κ is equivalent to λ_{G/H_1}. Let

$$\rho = \int_{G/H_1} \rho_{gH_1} \times \delta_{gH_1} d\kappa(\rho H_1)$$

be the desintegration of ρ. By (5.2),

$$\rho_{Q_H(g_1,g_2)gH_1} = \rho_{gH_1} \circ (S_1(g_1) \times S_2(g_2))$$

for κ-a.a. $gH_1 \in G/H_1$ and all $g_1, g_2 \in G$. Without loss of generality we may assume that this holds for all $g, g_1, g_2 \in G$. Let $\rho^* := \rho_{H_1}$. Since $Q_H(G \times \{1_G\})$ is transitive, we obtain that

$$\rho_{g_1 H_1} = \rho^* \circ (S_1(g_1) \times \mathrm{Id}) \quad \text{for all } g_1 \in G. \tag{7.2}$$

Moreover, $\rho^* \circ (S_1(g_1) \times S_2(g_2)) = \rho^*$ for all $(g_1, g_2) \in H$ since H is the Q_H-stabilizer of the point $H_1 \in G/H_1$. The converse is also true: every $S_1 \otimes S_2(H)$-invariant measure ρ^* gives rise to a measure $\rho \in \mathcal{P}((S_1 \otimes S_2) \times Q, \mathfrak{B}_Z, \kappa)$ by (7.2). ∎

The lemma below was formulated in [LeL] only in the Abelian case but the proof in the non-Abelian case remains unchanged. It also follows immediately from Proposition 6.1.

LEMMA 7.2: *Let G be amenable and let $\phi\colon X \to G$ be an ergodic cocycle of an ergodic measure preserving transformation T of $(X, \mathfrak{B}_X, \mu)$. Assume that S is a Borel G-action on $(Y, \mathfrak{B}_Y)$. Suppose that ρ is an ergodic $T_{\phi,S}$-invariant measure on $X \times Y$ whose marginal onto X equals μ. Then $\rho = \mu \times \nu$ for an ergodic S-invariant measure ν.*

The following theorem provides a full description for the ergodic self-joinings of $T_{\phi,S}$ when T has pure point spectrum and S is mildly mixing.

THEOREM 7.3: *Let T be an ergodic measure preserving transformation of the space $(X, \mathfrak{B}_X, \mu)$ with pure point spectrum and let $\eta \in J_2^e(T)$. Assume that S is a mildly mixing measure preserving action of G on $(Y, \mathfrak{B}_Y, \nu)$. Assume, moreover, that a cocycle $\phi\colon X \to G$ is ergodic. If the cocycle*

$$\phi \otimes \phi\colon X \times X \ni (x_1, x_2) \mapsto (\phi(x_1), \phi(x_2)) \in G^2$$

of $(X \times X, \mathfrak{B}_X \otimes \mathfrak{B}_X, \eta, T \times T)$ is regular and cohomologous to an ergodic cocycle ψ with values in some $H \in J_2(G)$, then there exists an affine isomorphism Λ of the simplex

$$J_2(T_{\phi,S}, \eta) := \{\eta' \in J_2(T_{\phi,S}) |\ \eta' \upharpoonright (\mathfrak{B}_X \otimes \mathfrak{B}_X) = \eta\}$$

onto the simplex of $S \otimes S(H)$-invariant measures on $Y \times Y$. More precisely, if

$$\phi \otimes \phi(x_1, x_2) = f(x_1, x_2)\psi(x_1, x_2) f(Tx_1, Tx_2)^{-1} \quad \eta\text{-a.e.}$$

for a measurable function $f\colon X^2 \to G^2$, we define a map $A\colon (X \times Y)^2 \to X^2 \times Y^2$ by setting

$$A(x_1, y_1, x_2, y_2) = (x_1, x_2, S \otimes S(f(x_1, x_2))(y_1, y_2)).$$

Then $\eta' \circ A^{-1} = \eta \times \Lambda(\eta')$ for all $\eta' \in J_2(T_{\phi,S}, \eta)$.

Otherwise, $J_2(T_{\phi,S}, \eta)$ consists of only one measure—the relatively independent extension of η.

Proof: Consider the first case. It has been studied in [LMN]. Though it was assumed that G is Abelian, this commutativity was not really used there. Therefore, we only briefly sketch the idea of the proof. Without loss of generality we may assume that $\phi \otimes \phi$ itself takes values in H. Indeed, changing a Rokhlin cocycle by a cohomologous one we always obtain an isomorphic extension. Then it remains to apply Lemma 7.2 and the first case easily follows.

Now we pass to the second case. Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ denote the $S\otimes S$-invariant sub-σ-algebras $\mathfrak{B}_Y \otimes \mathfrak{N}_Y$ and $\mathfrak{N}_Y \otimes \mathfrak{B}_Y$ of $\mathfrak{B}_Y \otimes \mathfrak{B}_Y$ respectively, where $\mathfrak{N}_Y$ stands for the trivial σ-algebra on Y. Since T has pure point spectrum, η is supported on the graph of a transformation $R \in C(T)$, i.e., $\eta(A \times B) = \mu(A \cap R^{-1}B)$ for all $A, B \in \mathfrak{B}_X$. Hence we may consider any measure $\eta' \in J_2(T_{\phi,S}, \eta)$ as a measure on $X \times Y \times Y$ invariant under $T_{\phi\times\phi\circ R, S\otimes S}$ and whose restriction to $\mathfrak{B}_X \otimes \mathfrak{L}_i$ is equal to $\mu \times \nu$, $i = 1, 2$. We have assumed that $\phi \odot \phi$ is either nonregular or $\phi\otimes\phi$ is regular but the corresponding group $H \notin J_2(G)$. Therefore this assumption, Proposition 4.2 and Lemma 3.1 imply that at least one of the coordinate actions $W_{\phi\times\phi\circ R}(G \times \{1_G\})$ or $W_{\phi\times\phi\circ R}(\{1_G\} \times G)$ is not of type I. It follows from Remark 6.2 that the affine isomorphism

$$\Lambda\colon \mathcal{P}(T_{\phi,\phi\circ R,S\otimes S}, \mathfrak{B}_X, \mu) \to \mathcal{P}(W_{\phi\times\phi\circ R}, \mathfrak{B}_{\Omega_{\phi\times\phi\circ R}}, \nu_{\phi\times\phi\circ R})$$

has the property that $\Lambda(\eta') \restriction (\mathfrak{B}_{\Omega_{\phi\times\phi\circ R}} \otimes \mathfrak{L}_i) = \nu_{\phi\times\phi\circ R} \times \nu$ for $i = 1, 2$. We can now apply Lemma 7.1(i) to conclude that the set

$$\{\rho \in \mathcal{P}(W_{\phi\times\phi\circ R} \times (S \odot S), \mathfrak{B}_{\Omega_{\phi\times\phi\circ R}}, \nu_{\phi\times\phi\circ R}) | \ \rho \restriction (\mathfrak{B}_{\Omega_{\phi\times\phi\circ R}} \otimes \mathfrak{L}_i) \\ = \nu_{\phi\times\phi\circ R} \times \nu, \ i = 1, 2\}$$

is a singleton. Hence the set

$$\mathcal{Q} := \{\eta' \in J_2(T_{\phi,S}) | \ \eta' \restriction (\mathfrak{B}_X \otimes \mathfrak{L}_i) = \mu \times \nu, \ i = 1, 2\}$$

is a singleton as well. It remains to notice that the relatively independent extension of η belongs to $\mathcal{Q}$. ∎

Remark 7.4: It is worthwhile to note that the second case in Theorem 7.3 with nonregular $\phi \times \phi$ (which was not considered in [LMN]) is not vacuous. Actually, let T and S be as above and V any nonsingular G^2-action such that the G-actions $V(\{1_G\} \times G)$ and $V(G \times \{1_G\})$ are both ergodic. Suppose that at least one of the latter two actions is properly ergodic. Next, fix a transformation $R \in C(T)$ such that the joint $\mathbb{Z}^2$-action with generators T and R is free (notice that such a transformation always exists since T has pure point spectrum). Denote by η the self-joining of T supported by the graph of R. Then by Proposition 4.3 and Theorem 7.3 there exists an ergodic cocycle $\phi \in Z^1(T, G)$ such that $J_2(T_{\phi,S}, \eta)$ is a singleton and the Mackey action associated to the cocycle $\phi \odot \phi$ of $(X \times X, \mathfrak{B}_X \otimes \mathfrak{B}_X, \eta, T \times T)$ is isomorphic to V.

8. Multipliers of $\mathcal{W}^{\perp}$

In this section the actions T, R, V and S considered below are assumed to be measure preserving. We need an auxiliary lemma from [LeP].

LEMMA 8.1 ([LeP, Proposition 5.1]): *Let T and R be ergodic transformations. If R is weakly mixing and $R \times R$ is disjoint from any ergodic self-joining of T, then $T \in \mathcal{M}(\{R\}^{\perp})$.*

It follows immediately that in order to prove that $T \in \mathcal{M}(\mathcal{W}^{\perp})$ it is enough to show that every ergodic self-joining of T is disjoint from $\mathcal{W}$.

Let T be an ergodic transformation on $(X, \mathfrak{B}_X, \mu)$ such that $T \in \mathcal{W}^{\perp}$. Let $\phi\colon X \to G$ be an ergodic cocycle of T and let S be an ergodic action of G on $(Y, \mathfrak{B}_Y, \nu)$. Assume that V is a weakly mixing transformation on $(Z, \mathfrak{B}_Z, \kappa)$. We claim that if $e(T_\phi)$ is countable then $T_{\phi,S} \perp V$. To prove this claim we notice first of all that $T \perp V$. Then observe that the cocycle $\phi \otimes 1 \in Z^1(T \times V, G)$ is ergodic. Indeed, the skew product extension

$$(T \times V)_{\phi\otimes 1} = T_\phi \times V \in \mathrm{Aut}(X \times G \times Z, \mu \times \lambda_G \times \kappa)$$

is ergodic if and only if $\sigma_V(e(T_\phi)) = 0$ (see [Aa, p. 81]), where σ_V denotes the measure of maximal spectral type of V on $L^2(Z, \kappa) \ominus \mathbb{C}1$. It suffices now to notice that σ_V is continuous and $e(T_\phi)$ countable. In view of Lemma 7.2, our claim follows. Thus we have proved the following.

PROPOSITION 8.2: *If T, ϕ, S are as above and $e(T_\phi)$ is countable, then $T_{\phi,S} \in \mathcal{W}^{\perp}$.*

Now we are ready to prove the main result of the paper, i.e., Theorem 0.1 stated in Introduction.

Proof of Theorem 0.1: Let η be an ergodic self-joining of $T_{\phi,S}$. Take a weakly mixing transformation V of a standard probability space $(Z, \mathfrak{B}_Z, \kappa)$. Consider a joining $\eta' \in J^e(T_{\phi,S}, T_{\phi,S}, V)$ projecting onto η. In view of Lemma 8.1, to prove the theorem it is enough to show that $\eta' = \eta \times \kappa$.

Since T has pure point spectrum, the projection of η onto $X \times X$ is supported by the graph of a transformation $R \in C(T)$. Hence we can consider η and η' as measures on $X \times Y \times Y$ and $X \times Y \times Y \times Z$ invariant under the transformations $T_{\phi\times\phi\circ R, S\otimes S}$ and $T_{\phi\times\phi\circ R, S\otimes S} \times V$ respectively. Since T and V are disjoint, the projection of η' onto $X \times Z$ is $\mu \times \kappa$. Moreover, $R' := R \times \mathrm{Id} \in C(T \times V)$ and we can rewrite $T_{\phi\times\phi\circ R, S\otimes S} \times V$ as $(T \times V)_{\phi\otimes 1 \times (\phi\otimes 1)\circ R', S\otimes S}$. Thus η' belongs to the simplex

$$\mathcal{P}((T \times V)_{\phi\otimes 1\times(\phi\otimes 1)\circ R', S\otimes S}, \mathfrak{B}_X \otimes \mathfrak{B}_Z, \mu \times \kappa). \tag{8.1}$$

Moreover, by Proposition 8.2,

$$\begin{aligned} \eta' \restriction (\mathfrak{B}_X \otimes \mathfrak{B}_Z \otimes \mathfrak{B}_Y \otimes \mathfrak{N}_Y) &= \mu \times \kappa \times \nu \quad \text{and} \\ \eta' \restriction (\mathfrak{B}_X \otimes \mathfrak{B}_Z \otimes \mathfrak{N}_Y \otimes \mathfrak{B}_Y) &= \mu \times \kappa \times \nu. \end{aligned} \tag{8.2}$$

Let $W_{\phi \times \phi \circ R}$ and $W_{(\phi\otimes 1)\times(\phi\otimes 1)\circ R'}$ act on their measure spaces $(\Omega, \mathfrak{B}_\Omega, \rho)$ and $(\Omega', \mathfrak{B}_{\Omega'}, \rho')$ respectively. By Proposition 6.1, the simplex (8.1) is affine isomorphic (via Λ) to the nonsingular simplex

$$\mathcal{P}(W_{(\phi\otimes 1)\times(\phi\otimes 1)\circ R'} \times (S \otimes S), \mathfrak{B}_{\Omega'}, \rho'). \tag{8.3}$$

Furthermore, in view of (8.2) and Remark 6.2,

$$\begin{aligned} \Lambda(\eta') \restriction (\mathfrak{B}_{\Omega'} \otimes \mathfrak{B}_Y \otimes \mathfrak{N}_Y) &= \rho' \times \nu \quad \text{and} \\ \Lambda(\eta') \restriction (\mathfrak{B}_{\Omega'} \otimes \mathfrak{N}_Y \otimes \mathfrak{B}_Y) &= \rho' \times \nu. \end{aligned} \tag{8.4}$$

It follows from Proposition 4.2 and the fact that $\phi \otimes 1$ is ergodic (see the proof of Proposition 8.2) that the G-actions

$$W_{\phi\otimes 1\times(\phi\otimes 1)\circ R'}(G \times \{1_G\}) \quad \text{and} \quad W_{\phi\otimes 1\times(\phi\otimes 1)\circ R'}(\{1_G\} \times G)$$

are ergodic. If at least one of them is properly ergodic, then by Lemma 7.1(i), there is only one measure satisfying (8.4) and belonging to the simplex (8.3). Hence there is only one measure satisfying (8.2) and belonging to the simplex (8.1). Since the measure $\eta \times \kappa$ satisfies these properties, we conclude that $\eta' = \eta \times \kappa$.

Consider now the case where the transformation groups

$$W_{\phi\otimes 1\times(\phi\otimes 1)\circ R'}(G \times \{1_G\}) \quad \text{and} \quad W_{\phi\otimes 1\times(\phi\otimes 1)\circ R'}(\{1_G\} \times G)$$

are both of type I. By Lemma 7.1(ii), there exist $H \in J_2(G)$ and a measure ρ^* on $(Y \times Y, \mathfrak{B}_Y \otimes \mathfrak{B}_Y)$ invariant under $S \otimes S(H)$ such that (up to isomorphism) $\Omega' = G/H_1$, $\rho' \sim \lambda_{G/H_1}$, $W_{\phi\otimes 1\times(\phi\otimes 1)\circ R'} = Q_H$ and

$$\Lambda(\eta') = \int_{\Omega'} \rho^* \circ (S(g) \times \mathrm{Id}) \times \delta_{gH_1} d\rho'(gH_1).$$

It follows from (8.4) that the marginals of ρ^* are equal to ν. Clearly,

$$H \supset (H_1 \times \{1_G\}) \cup (\{1_G\} \times H_2).$$

If H_1 is nontrivial, then it is noncompact by the assumption on G. Since S is mildly mixing, the transformation group $S(H_1)$ is also mildly mixing and, in particular, ergodic. Therefore by Lemma 5.4, ρ^* splits into a direct product $\nu \times \nu_1$. Clearly, $\nu_1 = \nu$ by our observation on the marginals of ρ^*. In a similar way, if H_2 is nontrivial then $\rho^* = \nu \times \nu$. Thus in both cases there exists only one measure satisfying (8.4) and belonging to the simplex (8.3). Thus we get again $\eta' = \eta \times \kappa$.

It remains to consider the case where $H_1 = H_2 = \{1_G\}$. Then the subset of measures satisfying (8.4) and belonging to (8.3) does not need to be a singleton. (Consider, for instance, the case where H is the diagonal subgroup of $G \times G$. Then the measure $\rho' \times \xi$ satisfies the two properties for any self-joining ξ of S.) To settle this case consider the natural projection $(\Omega', \mathfrak{B}_{\Omega'}, \rho') \to (\Omega, \mathfrak{B}_\Omega, \rho)$ intertwining $W_{(\phi\otimes 1)\times(\phi\otimes 1)\circ R'}$ with $W_{\phi\times\phi\circ R}$. By Proposition 5.6(ii) (the cocycle $\phi \times \phi \circ R$ plays now the role of ϕ from that corollary), it yields the ergodic decomposition of a transformation

$$D \in C(W_{(\phi\otimes 1)\times(\phi\otimes 1)\circ R'}) = C(Q_H).$$

Since $C(Q_H)$ is just the center $Z(G)$ of G acting on G by translations, we can identify D with an element $d \in Z(G)$. Let $K := \overline{\{d^n \mid n \in \mathbb{Z}\}}$. It is well known that the the quotient map $G \to G/K$ yields the ergodic decomposition of D. Any monothetic locally compact group is either compact or infinite discrete (and hence isomorphic to $\mathbb{Z}$) [HR]. Since D is conservative by Proposition 5.6(i), the latter is impossible for K. Hence K is compact and therefore trivial by our assumption on G. Thus the natural projection $\Omega' \to \Omega$ is the identity. Hence the natural projection of (8.3) onto the simplex $\mathcal{P}(W_{\phi\times\phi\circ R} \times (S \otimes S), \mathfrak{B}_\Omega, \rho)$ is one-to-one. Therefore, so is the natural projection of (8.1) onto

$$\mathcal{P}(T_{\phi\times\phi\circ R}, \mathfrak{B}_X \otimes \mathfrak{B}_Z, \mu \times \kappa)$$

(see Remark 6.3). Thus we get again $\eta' = \eta \times \kappa$. ∎

PROPOSITION 8.3: *Let G be amenable and let T be an ergodic transformation. Assume that there exists $R \in C(T) \setminus \{T^n \mid n \in \mathbb{Z}\}$. Then the subset*

$$\mathcal{L} := \{\phi \in Z^1(T,G) \mid \phi \text{ is ergodic and } e(T_\phi) = e(T)\}$$

is generic in $Z^1(T,G)$.

Proof: It follows from the proof of Theorem 4.2(i) from [Da1] that the subset

$$\mathcal{M} := \{\phi \in Z^1(T,G) \mid \phi \times \phi \circ R \text{ is ergodic}\}$$

is a dense G_δ in $Z^1(T,G)$. Next, if $\lambda \in e(T_\phi) \setminus e(T)$, then by [ALV] there exists a nontrivial continuous homomorphism (character) $\chi: G \to \mathbb{T}$ such that $\chi \circ \phi \approx \lambda$ in $Z^1(T,\mathbb{T})$. Since R commutes with T, we obtain $\chi \circ \phi \circ R \approx \lambda$ as well. Therefore, the cocycle $(\chi \times \chi) \circ (\phi \times \phi \circ R)$ is cohomologous to a constant (λ, λ) in $Z^1(T, \mathbb{T} \times \mathbb{T})$. Since the group generated by this constant is not dense in $\mathbb{T} \times \mathbb{T}$, we obtain that $(\chi \times \chi) \circ (\phi \times \phi \circ R)$ is not ergodic. Hence $\phi \times \phi \circ R$ is not ergodic as well. Thus $\mathcal{L} \supset \mathcal{M}$ and we are done. ∎

COROLLARY 8.4: *Let G, T and S be as in Theorem 0.1. Then for a generic cocycle $\phi \in Z^1(T,G)$ we have $T_{\phi,S} \in \mathcal{M}(\mathcal{W}^{\perp}) \setminus \mathcal{D}$.*

Proof: Since T has pure point spectrum, the centralizer $C(T)$ is nontrivial. Moreover, $e(T)$ is countable since for the probability preserving transformations the L^∞-spectrum equals the L^2-spectrum. It now follows from Theorem 0.1 and Proposition 8.2 that $T_{\phi,S} \in \mathcal{M}(\mathcal{W}^{\perp})$. It follows from Lemma 9.1 below that the extension $T_{\phi,S} \to T$ is relatively weakly mixing. Then by [Fu2], $T_{\phi,S}$ is not distal. ∎

Now we show how to deduce from that the main results of [Gl1]. Let $G = \mathbb{R}$ and S a horocycle flow corresponding to a lattice Γ in $\mathrm{PSL}_2(\mathbb{R})$. Recall that S is mixing of all degrees [Ma]. Let $(X, \mathfrak{B}, \mu) = (\mathbb{T}, \mathfrak{B}_{\mathbb{T}}, \lambda_{\mathbb{T}})$ and $Tx = xe^{2\pi i\alpha}$, $x \in \mathbb{T}$ ($\mathbb{T}$ denotes the circle group), for an irrational number $\alpha \in (0,1)$. Denote by $[0 : a_1, a_2, \ldots]$ the continued fraction expansion of α. Let $(q_n)_{n \geq 0}$ stand for the sequence of denominators of α, i.e.,

$$q_0 = 1, \quad q_1 = a_1, \quad q_{k+1} = a_{k+1} q_k + q_{k-1}, \quad k \geq 1.$$

We define a cocycle $\phi_0 \in Z^1(T, \mathbb{R})$ by setting $\phi_0(e^{2\pi it}) = t - 0.5$, where $0 \leq t < 1$. Ergodicity of ϕ_0 was established, e.g., in [Pa]. We need a stronger result.

PROPOSITION 8.5: *There exists a transformation $R \in C(T)$ such that the cocycle $\phi_0 \times \phi_0 \circ R$ of T is ergodic.*

To prove this proposition we need an auxiliary fact from [LMN] (see the proof of Lemma 3 in [LMN]).

LEMMA 8.6: *Given $\beta \in (0,1)$, let $Rx := xe^{2\pi i\beta}$, $x \in \mathbb{T}$. If the sequence $(\{q_n\beta\})_n$ has infinitely many accumulation points, then the cocycle $\phi_0 \times \phi_0 \circ R$ of T is ergodic.*

Proof of Proposition 8.5: Fix a sequence of positive reals $\epsilon_n \to 0$. Let $c_k \in (0,1)$ be a sequence of reals which contains every rational from $(0,1)$ infinitely many times. Then it is easy to select a sequence of positive integers l_k and a subsequence (q_{n_k}) of (q_n) such that the segments

$$I_k := \Big[\frac{l_k + c_k - \epsilon_k}{q_{n_k}}, \frac{l_k + c_k + \epsilon_k}{q_{n_k}}\Big]$$

form a nested sequence, i.e., $I_1 \supset I_2 \supset \cdots$. (Indeed it is suffices to notice that the distance between $[\frac{l+c_k-\epsilon_k}{q_n}, \frac{l+c_k+\epsilon_k}{q_n}]$ and $[\frac{l+1+c_k-\epsilon_k}{q_n}, \frac{l+1+c_k+\epsilon_k}{q_n}]$ tends to zero uniformly in l as $n \to \infty$ for each k.) Take $\beta \in \bigcap_{k=1}^{\infty} I_k$. Then $|\{q_{n_k}\beta\} - c_k| < \epsilon_k$ for all $k > 0$. Hence the sequence $\{q_n\beta\}$ has infinitely many accumulation points. Now we apply Lemma 8.6 and the result follows. ∎

We also note that there exist ergodic cocycles ϕ of an irrational rotation T such that $\phi \times \phi \circ R$ is not ergodic for any $R \in C(T)$ (an example of such a cocycle is given in [LMN] for $G = \mathbb{Z}$).

Remark 8.7: Let us also notice that using the a.a.c.c.p. method from [KwLR] one can construct smooth (even analytic) real valued cocycles ϕ (over irrational rotations under some Diophantine restrictions) satisfying the assertion of Proposition 8.6. Now, by putting a horocycle flow on the fiber we will obtain examples of nondistal smooth multipliers of $\mathcal{W}^{\perp}$.

Let Φ_0 stand for the family of continuous cocycles of T with zero mean. Endowed with the topology of uniform convergence Φ_0 is a Polish space. Since T is uniquely ergodic, we have

$$\Phi_0 = \overline{\{f - f \circ T \mid f \colon \mathbb{T} \to \mathbb{R} \text{ is continuous}\}}.$$

By [Ko] (see also [Ru]), ϕ_0 is cohomologous to a cocycle $\psi \in \Phi_0$. Then, of course, the set

$$\{f + \psi - f \circ T \mid \text{for all continuous } f \colon \mathbb{T} \to \mathbb{R}\}$$

is dense in Φ_0. It is also a subset of $\mathcal{M}$. Since the uniform topology is stronger than the topology of convergence in measure and $\mathcal{M}$ is a G_δ in $Z^1(X, \mathbb{R})$, we conclude that $\Phi_0 \cap \mathcal{M}$ is a dense G_δ in Φ_0. Thus we have proved an extension of the most technically involved statement in [Gl1]—Theorem 5.1 (proved there under some Diophantine restrictions on α):

PROPOSITION 8.8: *For any irrational number α, the subset*

$$\{\phi \in \Phi_0 \mid T_\phi \text{ is ergodic and } e(T_\phi) = e(T)\}$$

is generic in Φ_0.

The corollary below follows from this and Theorem 0.1.

COROLLARY 8.9: *For every ϕ from a dense G_δ-subset of Φ_0, the strictly ergodic homeomorphism $T_{\phi,S}$ of the compact manifold $X \times Y$ is in $\mathcal{M}(\mathcal{W}^\perp)$ but not in $\mathcal{D}$.*

This extends [Gl1, Theorem 4.1] where it was assumed additionally that Γ is maximal and nonarithmetic and α is rather special.

9. Semisimple extensions of transformations with pure point spectrum

We first extend an assertion on relative weak mixing from [LeL], where it was assumed that G is Abelian and spectral theory was used in the proof.

LEMMA 9.1: *Let T be a measure preserving transformation and let $\phi: X \to G$ be a cocycle of T. Assume that S is a mildly mixing G-action. If $T_{\phi,S}$ is ergodic then the extension $T_{\phi,S} \to T$ is relatively weakly mixing.*

Proof: What we need in fact to prove is that the transformation $T_{\phi,S\times S}$ of the space $(X \times Y \times Y, \mathfrak{B}_X \otimes \mathfrak{B}_Y \otimes \mathfrak{B}_Y, \mu \times \nu \times \nu)$ is ergodic. By Proposition 6.1, the measure $\mu \times \nu \times \nu$ corresponds under an affine map to the measure

$$\nu_\phi \times \nu \times \nu \in \mathcal{P}(W_\phi \times S \times S, \mathfrak{B}_{\Omega_\phi}, \nu_\phi).$$

Suppose first that W_ϕ is properly ergodic. Since $S \times S$ is mildly mixing, we conclude that $\nu_\phi \times \nu \times \nu$ is ergodic for $W_\phi \times S \times S$. Hence $\mu \times \nu \times \nu$ is ergodic for $T_{\phi,S\times S}$.

Now let W_ϕ be of type I. This means that the cocycle ϕ is cohomologous to an ergodic cocycle with values in a closed subgroup H of G. Without loss of generality we may assume that ϕ itself enjoys this property (changing ϕ with a cohomologous cocycle we obtain an isomorphic extension). Since $T_{\phi,S} = T_{\phi,S(H)}$ is ergodic, so is $S(H)$. If H were compact then $S(H)$ and hence $S(G)$ would be of type I. That contradicts the mild mixing assumption on S. Hence H is not compact and therefore $S(H)$ is mildly mixing. Now ϕ is ergodic (as a cocycle with values in H), so W_ϕ is trivial and, since $(S \times S)(H)$ is ergodic, we are done. ∎

Definition 9.2: A probability preserving action S of a l.c.s.c. group G on $(Y, \mathfrak{B}_Y, \nu)$ is called **2-fold-extra-simple** if for any continuous group automorphism $\theta: G \to G$, every ergodic joining of S and $S \circ \theta$ is either the product $\nu \times \nu$ or a joining supported by the graph of a transformation $R \in \mathrm{Aut}_0(Y, \nu)$ such that $RS(g)R^{-1} = S(\theta(g))$ for all $g \in G$.

Notice that a 2-fold simple action S is 2-fold-extra-simple if and only if for any continuous group automorphism θ: $G \to G$, the G-action $S \circ \theta$ is either isomorphic to S or disjoint from it. Suppose that the center of G has no compact subgroups. If S is simple and prime (in particular, if it has the MSJ property, see [JRu, Theorem 3.1]) then S is 2-fold-extra-simple by [JRu, Corollary 4.3].

For example, if $G = \mathbb{R}$, the horocycle flow corresponding to a maximal non-arithmetic lattice $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ and the Chacon flow are 2-fold-extra-simple since they have the MSJ property by [Rat] and [JPa] respectively.

Example 9.3 (simple but not 2-fold-extra-simple transformation)*:* Let K be a compact metric group. Suppose that T has the MSJ property and T and T^{-1} are conjugate via a transformation $R \in \mathrm{Aut}_0(X, \mu)$ (see [JRS] for examples of such maps). Denote by $\mathcal{R}$ the T-orbit equivalence relation. It is easy to see that $R \in N[\mathcal{R}] \setminus [\mathcal{R}]$. From the proof of [Da1, Theorem 4.2(i)] we deduce that the cocycles $\phi \in Z^1(T, K)$ such that $\alpha_\phi \times \alpha_\phi \circ R$ is ergodic form a dense G_δ subset of $Z^1(T, K)$. Recall that $\alpha_\phi(Tx, x) = \phi(x)$ for a.a. x (see Section 1). Fix such a ϕ. Next, as in the proof of Proposition 8.3 one can check that $e(T_\phi) = e(T)$. Since $e(T_\phi)$ and $e(T)$ are equal to the L^2-spectrum of T_ϕ and T respectively and T is weakly mixing, T_ϕ is also weakly mixing. Then by [JRu, Theorem 5.4], T_ϕ is simple. We claim that it is not 2-fold-extra-simple. Indeed, assume that the contrary holds. Since $T_\phi \not\perp (T_\phi)^{-1}$ (these transformations have a common factor—T), there exists a transformation $S' \in \mathrm{Aut}_0(X \times K, \mu \times \lambda_K)$ which conjugates T_ϕ and $(T_\phi)^{-1}$. Then by [GJLR, Theorem 5], there exists a transformation S of $(X, \mathfrak{B}_X, \mu)$ such that $S'(x, k) = (Sx, S_2(x, k))$ for a.a. $(x, k) \in X \times K$. (Though it was assumed in [GJLR] that K is commutative, the proof of the cited fact holds for noncommutative groups as well.) Clearly, S conjugates T and T^{-1}. Hence

$$SR^{-1} \in C(T) = \{T^n | \, n \in \mathbb{Z}\}$$

and therefore $\alpha_\phi \circ S \approx \alpha_\phi \circ R$. Moreover, by [GJLR, Proposition 7] (Abelian case) and [Da1, Theorem 5.3] (general case), there is a group automorphism l of K such that $\alpha_\phi \circ S \approx l \circ \alpha_\phi$. Thus the cocycle $l \circ \alpha_\phi \times \alpha_\phi \circ R$ is cohomologous to $l \circ \alpha_\phi \times \alpha_\phi \circ S$, which is in turn cohomologous to the cocycle $l \circ \alpha_\phi \times l \circ \alpha_\phi$ taking values in the diagonal subgroup of G^2. Hence it is never ergodic. Since the ergodicity of a cocycle is invariant under composition with a group automorphism, it follows that $\alpha_\phi \times \alpha_\phi \circ R$ is neither ergodic, a contradiction.

Now we are ready to give a proof of Theorem 0.2 stated in Introduction.

Proof of Theorem 0.2: Let η be any ergodic self-joining of $T_{\phi,S}$. As in the proof of Theorem 7.3, we may consider η as an ergodic $T_{\phi\times\phi\circ R,S\otimes S}$-invariant measure on $X \times Y \times Y$ such that

$$(9.1)\quad \eta \restriction (\mathfrak{B}_X \otimes \mathfrak{B}_Y \otimes \mathfrak{N}_Y) = \mu \times \nu \quad \text{and} \quad \eta \restriction (\mathfrak{B}_X \otimes \mathfrak{N}_Y \otimes \mathfrak{B}_Y) = \mu \times \nu,$$

where R is a transformation from $C(T)$. Suppose first that the cocycle $\phi \times \phi \circ R$ is not regular or is regular but cohomologous to an ergodic cocycle with values in a closed subgroup $H \notin J_2(G)$. Then $\eta = \mu \times \nu \times \nu$ by Theorem 7.3. It follows from Lemma 9.1 that the extension

$$(9.2)\qquad (T_{\phi\times\phi\circ R,S\otimes S}, \eta) = ((T_{\phi,S})_{(\phi\circ R)\otimes 1}, \mu \times \nu \times \nu) \to (T_{\phi,S}, \mu \times \nu)$$

is relatively weakly mixing and we are done.

In the remaining case we may assume that $\phi \times \phi \circ R$ is ergodic itself as a cocycle with values in $H \in J_2(G)$. By Theorem 7.3, $\eta = \mu \times \rho^*$, where ρ^* is an $S \otimes S(H)$-invariant measure. It follows from (9.1) that the marginals of ρ^* are both equal to ν. Arguing as in the proof of Theorem 0.1, we obtain that $\rho^* = \nu \times \nu$ whenever $H \cap (\{1_G\} \times G)$ or $H \cap (G \times \{1_G\})$ is nontrivial. Thus we come to the case considered above.

Finally, let H be the graph of a group automorphism θ: $G \to G$. Since S is 2-fold-extra-simple, either $\rho^* = \nu \times \nu$ or ρ^* is supported by the graph of some ν-preserving transformation Q such that $QS(g)Q^{-1} = S(\theta(g))$ for all $g \in G$. In both cases (9.2) is relatively weakly mixing. Summarizing all the cases we see that $T_{\phi,S}$ is semisimple.

The relative weak mixing of $T_{\phi,S} \to T$ has been established in Lemma 9.1. ∎

Notice that if $\phi \circ R \not\approx \theta \circ \phi$ for all $R \in C(T)$ and nontrivial group automorphisms θ, then we can replace (relax) the condition of 2-fold-extra-simplicity in Theorem 0.2 with the 2-fold-simplicity.

References

[Aa] J. Aaronson, *An Introduction to Infinite Ergodic Theory*, Mathematical Surveys and Monographs, Vol. 50, American Mathematical Society, Providence, RI, 1997.

[ALV] J. Aaronson, M. Lemańczyk and D. Volný, *A cut salad of cocycles*, Fundamenta Mathematicae **157** (1998), 99–119.

[AbR] L. M. Abramov and V. A. Rokhlin, *The entropy of a skew product of measure preserving transformation*, American Mathematical Society Translations **48** (1965), 225–245.

[An] H. Anzai, *Mixing up properties of Brownian motion*, Osaka Journal of Mathematics **1** (1950), 51–58.

[CK] A. Connes and W. Krieger, *Measure space automorphisms, the normalizers of their full groups, and approximate finiteness*, Journal of Functional Analysis **24** (1977), 336–352.

[Da1] A. I. Danilenko, *Comparison of cocycles of measured equivalence relation and lifting problems*, Ergodic Theory and Dynamical Systems **18** (1998), 125–151.

[Da2] A. I. Danilenko, *Quasinormal subrelations of ergodic equivalence relations*, Proceedings of the American Mathematical Society **126** (1998), 3361–3370.

[FM] J. Feldman and C. C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras*. I, Transactions of the American Mathematical Society **234** (1977), 289–324.

[Fu1] H. Furstenberg, *Disjointness in ergodic theory, minimal sets and diophantine approximation*, Mathematical Systems Theory **1** (1967), 1–49.

[Fu2] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton University Press, 1981.

[FuW] H. Furstenberg and B. Weiss, *The finite multipliers of infinite ergodic transformations*, Lecture Notes in Mathematics **668** (1978), 127–132.

[Gl1] E. Glasner, *On the multipliers of $\mathcal{W}^{\perp}$*, Ergodic Theory and Dynamical Systems **14** (1994), 129–140.

[Gl2] E. Glasner, *Ergodic theory via joinings*, Mathematical Surveys and Monographs, Vol. 101, American Mathematical Society, Providence, RI, 2003.

[GlW] E. Glasner and B. Weiss, *Processes disjoint from weak mixing*, Transactions of the American Mathematical Society **316** (1989), 689–703.

[GS1] V. Ya. Golodets and S. D. Sinelshchikov, *Amenable ergodic actions of groups and images of cocycles*, Soviet Mathematics Doklady **41** (1990), 523–525.

[GS2] V. Ya. Golodets and S. D. Sinelshchikov, *Classification and structure of cocycles of amenable ergodic equivalence relations*, Journal of Functional Analysis **121** (1994), 455–485.

[GJLR] G. R. Goodson, A. del Junco, M. Lemańczyk and D. J. Rudolph, *Ergodic transformations conjugate to their inverses by involutions*, Ergodic Theory and Dynamical Systems **16** (1996), 97–124.

[GrS] G. Greschonig and K. Schmidt, *Ergodic decomposition of quasi-invariant probability measures*, Colloquium Mathematicum **84/85** (2000), 495–514.

[HR] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis,*, Vol. I, Springer-Verlag, Berlin–Göttingen–Heidelberg, 1963.

[JRu] A. del Junco and D. Rudolph, *On ergodic actions whose self-joinings are graphs,* Ergodic Theory and Dynamical Systems **7** (1987), 531–557.

[JLM] A. del Junco, M. Lemańczyk, and M. Mentzen, *Semisimplicity, joinings, and group extensions,* Studia Mathematica **112** (1995), 141–164.

[JPa] A. del Junco and K. K. Park, *An example of a measure-preserving flow with minimal self-joinings,* Journal d'Analyse Mathématique **42** (1982/83), 199–209.

[JRS] A. del Junco, M. Rahe and L. Swanson, *Chacon's automorphism has minimal self-joinings,* Journal d'Analyse Mathématique **37** (1980), 276–284.

[Ki] E. Kin, *Skew products of dynamical systems,* Transactions of the American Mathematical Society **166** (1972), 27–43.

[Ko] A. V. Kochergin, *On the homology of function over dynamical systems,* Doklady Akademii Nauk SSSR **231** (1976), 795–798.

[KwLR] J. Kwiatkowski, M. Lemańczyk and D. Rudolph, *A class of real valued cocycles having an analytic coboundary modification,* Israel Journal of Mathematics **87** (1994), 337–360.

[LeL] M. Lemańczyk and E. Lesigne, *Ergodicity of Rokhlin cocycles,* Journal d'Analyse Mathématique **85** (2001), 43–86.

[LeM] M. Lemańczyk and M. K. Mentzen, *Compact subgroups in the centralizer of natural factors of an ergodic group extension of a rotation determine all factors,* Ergodic Theory and Dynamical Systems **10** (1990), 763–776.

[LMN] M. Lemańczyk, M. K. Mentzen and H. Nakada, *Semisimple extensions of irrational rotations,* Studia Mathematica **156** (2003), 31–57.

[LeP] M. Lemańczyk and F. Parreau, *Rokhlin extensions and lifting disjointness,* Ergodic Theory and Dynamical Systems **23** (2003), 1525–1550.

[Ma] B. Marcus, *The horocycle flow is mixing of all degrees,* Inventiones Mathematicae **46** (1978), 201–209.

[Me] M. K. Mentzen, *Ergodic properties of group extensions of dynamical systems with discrete spectra,* Studia Mathematica **101** (1991), 20–31.

[Pa] D. Pask, *Skew products over irrational rotation,* Israel Journal of Mathematics **69** (1996), 65–74.

[Ra] A. Ramsay, *Virtual groups and group actions,* Advances in Mathematics **6** (1971), 253–322.

[Rat] M. Ratner, *Horocycle flows, joinings and rigidity of products,* Annals of Mathematics **118** (1983), 277–313.

[Ro] E. A. Robinson, *A general condition for lifting theorems,* Transactions of the American Mathematical Society **330** (1992), 725–755.

[Ru] D. Rudolph, *$\mathbb{Z}^n$ and $\mathbb{R}^n$ cocycle extension and complementary algebras,* Ergodic Theory and Dynamical Systems **6** (1986), 583–599.

[Sc] K. Schmidt, *Cocycles of ergodic transformation groups,* Lecture Notes in Mathematics, Vol. 1, McMillan Co of India, 1977.

[SWa] K. Schmidt and P. Walters, *Mildly mixing actions of locally compact groups,* Proceedings of the London Mathematical Society **45** (1982), 506–518.

[Th] J.-P. Thouvenot, *Some properties and applications of joinings on ergodic theory,* in *Ergodic Theory and its Connections with Harmonic Analysis,* London Mathematical Society Lecture Note Series 205, Cambridge University Press, 1995, pp. 207–235.